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# RULED SURFACES WITH ALTERNATIVE MOVING FRAME IN EUCLIDEAN 3-SPACE

**SOUKAINA OUARAB[1], AMINA OUAZZANI CHAHDI[2] AND MALIKA IZID[3]**
[(1,2,3)] Department of Mathematics and Computer Science,
Ben M'Sik Faculty of Sciences,
Hassan II University of Casablanca, Morocco

**Abstract**

In this paper, we construct and study a special ruled surface with the alternative moving frame of its base curve in euclidean 3-space. We investigate the main characteristic properties of that ruled surface and characterize it in terms of its Gaussian curvature, mean curvature and striction curve in some special cases. Moreover, we presente a study with illustrations of this kind of ruled surfaces in the case where are generated by some important general and slant helices.

## 1. Introduction

In differential geometry of curves and surfaces [11], a ruled surface represents one of the most fascinating topics in surface theory, it is defined by choosing a curve which called base curve and a line along that curve (ruling). An important number of researchers in many papers have studied one of the moving frames of its base curve. In [6] the authors have defined a family of ruled surfaces generated by some special curves using Frenet

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frame of those curves in euclidean 3-space, it has been precisely a study with general and slant helices. The general helix is the curve such that the tangent makes a constant angle with a fixed straight line which is called the axis of the general helix, it is characterized by a necessary and sufficient condition that a curve be a general helix is that the ratio $\frac{\tau}{\kappa}$ is constant along the curve, where $\kappa$ and $\tau$ denote the curvature and the torsion, respectively [12]. The slant helix is the curve such that the normal line makes a constant angle with a fixed straight line which is called the axis of the slant helix. Izumiya and Takeuchi [10] proved that: A curve is a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix $\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$ is constant along the curve. On an other hand, in [1] Frenet frame has been used to defined and investigate the principal normal vectors belonging to striction curves of Frenet and Bertrandian Frenet ruled surface. In [2] the authors have studied ruled surface with poinwise 1-type Gauss Map using the Darboux vector which is defined by the Frenet frame vectors.

However, many researchers have focused on the study of ruled surface with Darboux frame, effectively, in [5]the authors have defined the ruled surface with Darboux frame and invetigated some special characteristic properties of that surface and gave the relation between Darboux and Frenet frame. In [4] the parallel ruled surface with Darboux frame was introduced in euclidean 3-space, some characteristic properties such as developability, striction curve and distribution parameter of this type of surface was given in euclidean 3-space. Darboux frame was used also in [7] but in Minkowski 3-space where authors have given necessary and sufficient condition for a ruled surface to be developable.

In this paper we are inspired to investigate study of ruled surface with alternative moving frame which introduced in [3]. We defined the ruled surface whose rulings are linear combinations of alternative moving frame vectors of its base curve. We investigate the most important characteristic properties of that ruled surface, we give some characterizations in some special cases relatively to developability and minimality of the ruled surface. Moreover, we renforce our work by examples of this kind of ruled surfaces in the case where are generated by some important general and slant helices [8, 9].

## 2. Preliminaries

In this section, we will present some basic concepts related to ruled surface and aternative moving frame.

Let $E^3$ be a 3-dimensional euclidean space provided with the metric given by $\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2$, where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $E^3$.

Due to a regular curve $c(s)$ there exists the Frenet frame and it is denoted by $\{\overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B}\}$. In Frenet frame, $\overrightarrow{T}$ is the unit tangent vector of the curve $c(s)$, $\overrightarrow{N}$ is the unit normal vector and $\overrightarrow{B}$ is the binormal vector which is defined by $\overrightarrow{B} = \overrightarrow{T} \wedge \overrightarrow{N}$.

The derivative formulae of Frenet frame is given by:

$$\begin{bmatrix} \overrightarrow{T}'(s) \\ \overrightarrow{N}'(s) \\ \overrightarrow{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T}(s) \\ \overrightarrow{N}(s) \\ \overrightarrow{B}(s) \end{bmatrix},$$

where $\kappa$ is the curvature and $\tau$ is the torsion of $c(s)$ respectively.

On other hand, the alternative moving frame of a regular curve $c(s)$ for which the curvature $\kappa$ do not cancel each other is defined by the three vectors $\overrightarrow{N}$, $\overrightarrow{C} = \frac{\overrightarrow{N}'}{\left\|\overrightarrow{N}'\right\|}$ and the last $\overrightarrow{W} = \overrightarrow{N} \wedge \overrightarrow{C}$ which called Darboux vector. Derivative formulae are defined as follows:

$$\begin{bmatrix} \overrightarrow{N}'(s) \\ \overrightarrow{C}'(s) \\ \overrightarrow{W}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \alpha(s) & 0 \\ -\alpha(s) & 0 & \beta(s) \\ 0 & -\beta(s) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{N}(s) \\ \overrightarrow{C}(s) \\ \overrightarrow{W}(s) \end{bmatrix},$$

where $\alpha = \sqrt{\kappa^2 + \tau^2}$, $\beta = \sigma\alpha$ and $\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)'$.

A ruled surface is generated by a one-parameter family of straight lines and has a parametric representation:

$$\Psi(s, v) = c(s) + v\overrightarrow{X}(s), \qquad \left\|\overrightarrow{X}(s)\right\| = 1 \tag{1}$$

where $c(s)$ is called the base curve and $\overrightarrow{X}(s)$ is the unit represents a space curve which representing the direction of straight line (ruling).

If there exists a common perpendicular to two constructive rulings in the ruled surface, then the foot of the common perpendicular on the main rulings is called a central point. The locus of the central point is called striction curve. The parametrization of the striction curve $\mu$ on the ruled surface [1] is given by

$$\mu(s) = c(s) - \frac{\left\langle c'(s), \overrightarrow{X}'(s) \right\rangle}{\left\| \overrightarrow{X}'(s) \right\|^2} \overrightarrow{X}(s).$$

If $\left\| \overrightarrow{X}\prime(s) \right\| = 0$, then ruled surface does not have any striction curve. In this case it is called cylindrical ruled surface. Thus the base curve can be taken as a striction curve. Let $M = \Psi(s, v)$ be a regular point of the surface $\Psi$, the standard unit normal vector field on $\Psi$ at $M$ can be defined by:

$$\frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|},$$

where $\Psi_s = \frac{\partial \Psi(s,v)}{\partial s}$ and $\Psi_v = \frac{\partial \Psi(s,v)}{\partial v}$. The first $I$ and the second $II$ fundamental forms of the surface $\Psi$ relatively to the point $M$ are given by, respectively

$$\begin{aligned} I &= I\left(\Psi_s ds + \Psi_v dv\right) = E ds^2 + 2F ds dv + G dv^2, \\ II &= II\left(\Psi_s ds + \Psi_v dv\right) = e ds^2 + 2f ds dv + g dv^2, \end{aligned}$$

where

$$\begin{aligned} E &= \|\Psi_s\|^2, \ F = \langle \Psi_s, \Psi_v \rangle, \ G = \|\Psi_v\|^2 = 1, \\ e &= \left\langle \Psi_{ss}, \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|} \right\rangle, \ f = \left\langle \Psi_{sv}, \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|} \right\rangle, \ g = \left\langle \Psi_{vv}, \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|} \right\rangle = 0. \end{aligned}$$

On the other hand, Gaussian curvature $K$ and mean curvature $H$ are given as follows, respectively

$$K = \frac{eg - f^2}{EG - F^2} = -\frac{f^2}{E - F^2}, \quad H = \frac{Eg + Ge - 2Ff}{2\left(EG - F^2\right)} = \frac{e - 2Ff}{2\left(E - F^2\right)}.$$

Therefore, we present the following important definitions:

**Definition 2.1** : A regular surface is developable if and only if its Gaussian curvature vanishes identically.

**Definition 2.2** : A regular surface is minimal if and only if its mean curvature vanishes identically.

Normal curvature, geodesic curvature and geodesic torsion of the curve $c(s)$ on the surface $\Psi$ are defined by:

$$\rho_n = \left\langle \overrightarrow{U}, \overrightarrow{T}\prime \right\rangle, \ \rho_g = \left\langle \overrightarrow{U} \wedge \overrightarrow{T}, \overrightarrow{T}\prime \right\rangle, \ \theta_g = -\left\langle \overrightarrow{U} \wedge \overrightarrow{T}, \overrightarrow{U}\prime \right\rangle, \qquad (2)$$

where $\overrightarrow{U} = \overrightarrow{U}(s) = \dfrac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|}(s, 0)$ is the unit normal on the ruled surface $\Psi$ at a point of its base curve.

Now, we can write the following important definitions:

**Definition 2.3** : For a curve $c = c(s)$ lying on a surface $\Psi$ we have the following definitions:

1. $c(s)$ is an asymptotic line for $\Psi$ if and only if its normal curvature $\rho_n$ vanishes.
2. $c(s)$ is a geodesic curve for $\Psi$ if and only if its geodesic curvature $\rho_g$ vanishes.
3. $c(s)$ is a principal line for $\Psi$ if and only if its geodesic torsion $\theta_g$ vanishes.

## 3. Ruled Surface with Alternative Moving Frame

Let $c = c(s)$ be a regular curve whose curvature $\kappa$ do not cancel each other and denoting by $\{\overrightarrow{N}, \overrightarrow{C}, \overrightarrow{W}\}$ its alternative moving frame.

Derivative formulae of alternative moving frame of $c = c(s)$ are defined by

$$\begin{bmatrix} \overrightarrow{N}'(s) \\ \overrightarrow{C}'(s) \\ \overrightarrow{W}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \alpha(s) & 0 \\ -\alpha(s) & 0 & \beta(s) \\ 0 & -\beta(s) & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{N}(s) \\ \overrightarrow{C}(s) \\ \overrightarrow{W}(s) \end{bmatrix}, \tag{3}$$

where $\alpha = \sqrt{\kappa^2 + \tau^2}$, $\beta = \sigma\alpha$ and $\sigma = \dfrac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left(\dfrac{\tau}{\kappa}\right)'$.

Considering the ruled surface $\Psi$ defined as follows

$$\Psi : (s, v) \mapsto c(s) + v\left[x_1\overrightarrow{N}(s) + x_2\overrightarrow{C}(s) + x_3\overrightarrow{W}(s)\right], \tag{4}$$

where, $x_1$, $x_2$ and $x_3$ are three constants satisfying $x_1^2 + x_2^2 + x_3^2 = 1$.

Differentiating (4) with respect to $s$ and $v$ respectively and using alternative frame formulae (3), we get

$$\Psi_s = -vx_2\alpha\overrightarrow{N} + \left[-\frac{\kappa}{\alpha} + v(x_1\alpha - x_3\beta)\right]\overrightarrow{C} + \left[\frac{\tau}{\alpha} + vx_2\beta\right]\overrightarrow{W}, \tag{5}$$

$$\Psi_v = x_1\overrightarrow{N} + x_2\overrightarrow{C} + x_3\overrightarrow{W}, \tag{6}$$

then, the normal vector of the ruled surface (4) is

$$\Psi_s \wedge \Psi_v = [a_0 + va_1]\overrightarrow{N} + \left[\frac{\tau}{\alpha}x_1 + va_2\right]\overrightarrow{C} + \left[\frac{\kappa}{\alpha}x_1 + va_3\right]\overrightarrow{W}, \tag{7}$$

where

$$\begin{aligned} a_0 &= -\left(x_2\frac{\tau}{\alpha}+x_3\frac{\kappa}{\alpha}\right), \quad a_1 = x_1x_3\alpha-\left(x_2^2+x_3^2\right)\beta, \\ a_2 &= x_2\left(x_3\alpha+x_1\beta\right), \quad a_3 = x_1x_3\beta-\left(x_1^2+x_2^2\right)\alpha. \end{aligned}$$

Note that the condition of regularty of ruled surface (4) along its base curve is:

$$x_1^2+\left(x_2\frac{\tau}{\alpha}+x_3\frac{\kappa}{\alpha}\right)^2 \neq 0.$$

From (5) and (6), we obtain the components $E$, $F$ and $G$ of the first fundamental form at a regular point $M=\Psi(s,v)$, respectively

$$\begin{cases} E = 1+2v\left[-\kappa x_1+\left(x_3\kappa+x_2\tau\right)\dfrac{\beta}{\alpha}\right]+v^2\left[x_2^2\left(\alpha^2+\beta^2\right)+\left(x_1\alpha-x_3\beta\right)^2\right] \\ F = -x_2\dfrac{\kappa}{\alpha}+x_3\dfrac{\tau}{\alpha}, \\ G = 1, \end{cases} \tag{8}$$

differentiating (5) and (6), we get

$$\begin{cases} \Psi_{ss} = \left[\kappa-vA_1\right]\overrightarrow{N}+vA_2\overrightarrow{C}+vA_3\overrightarrow{W}, \\ \Psi_{vs} = -\alpha x_2\overrightarrow{N}+\left(\alpha x_1-\beta x_3\right)\overrightarrow{C}+\beta x_2\overrightarrow{W}, \\ \Psi_{vv} = 0, \end{cases} \tag{9}$$

then, from (7) and (9) the components $e$, $f$ and $g$ of the second fundamental form at a regular point $M=\Psi\left(s,v\right)$ are as follows, respectively

$$\begin{cases} e = \dfrac{\left[\kappa-vA_1\right]\left[a_0+va_1\right]+vA_2\left[\frac{\tau}{\alpha}x_1+va_2\right]+vA_3\left[\frac{\kappa}{\alpha}x_1+va_3\right]}{\sqrt{a_0^2+x_1^2+2v\left[a_0a_1+x_1\left(a_2\frac{\tau}{\alpha}+a_3\frac{\kappa}{\alpha}\right)\right]+v^2\left(a_1^2+a_2^2+a_3^2\right)}}, \\ f = \dfrac{\left(1-x_3^2-x_1x_3\sigma\right)\tau+x_2\kappa\left(x_3+x_1\sigma\right)}{\sqrt{a_0^2+x_1^2+2v\left[a_0a_1+x_1\left(a_2\frac{\tau}{\alpha}+a_3\frac{\kappa}{\alpha}\right)\right]+v^2\left(a_1^2+a_2^2+a_3^2\right)}}, \\ g = 0, \end{cases} \tag{10}$$

where

$$A_1 = x_2\alpha'+\alpha\left(x_1\alpha-x_3\beta\right), \; A_2 = -x_2\left(\alpha^2+\beta^2\right)+\left(x_1\alpha'-x_3\beta'\right), \; A_3 = \left(x_1\alpha-x_3\beta\right)\beta+x_2\beta'.$$

Thereafter, from (8) and (11), Gaussian and mean curvatures of the ruled surface $\Psi$

along its base curve take the following forms, respectively

$$\begin{cases} K(s,0) = -\left(\dfrac{\left(1 - x_3^2 - x_1 x_3 \sigma\right)\tau + x_2 \kappa\left(x_3 + x_1 \sigma\right)}{a_0^2 + x_1^2}\right)^2, \\ H(s,0) = -\dfrac{\kappa\left(x_2 \tau + x_3 \kappa\right) + 2\left(-x_2 \kappa + x_3 \tau\right)\left[\left(1 - x_3^2 - x_1 x_3 \sigma\right)\tau + x_2 \kappa\left(x_3 + x_1 \sigma\right)\right]}{2\alpha\left(a_0^2 + x_1^2\right)^{3/2}}. \end{cases}$$

On the other hand, supposing $\Psi$ non-cylindrical ruled surface ($\|X'(s)\| \neq 0, \forall s$), i.e.,

$\left\|\overrightarrow{X}'(s)\right\|^2 = x_2^2\left(\alpha^2 + \beta^2\right) + \left(\alpha x_1 - \beta x_3\right)^2 \neq 0$, where $\overrightarrow{X}(s) = x_1 \overrightarrow{N}(s) + x_2 \overrightarrow{C}(s) + x_3 \overrightarrow{W}(s)$,

then, striction curve of (4) denoted by $\mu$ takes the following form:

$$\mu(s) = c(s) - \frac{-\kappa\left(x_1 - \sigma x_3\right) + \tau\sigma x_2}{x_2^2\left(\alpha^2 + \beta^2\right) + \left(\alpha x_1 - \beta x_3\right)^2}\left[x_1 \overrightarrow{N}(s) + x_2 \overrightarrow{C}(s) + x_3 \overrightarrow{W}(s)\right].$$

The unit normal vector $\overrightarrow{U} = \overrightarrow{U}(s)$ of the ruled surface (4) at a point $(s,0)$ of its base curve is given by

$$\overrightarrow{U} = \overrightarrow{U}(s) = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|}(s,0) = \frac{1}{\sqrt{a_0^2 + x_1^2}}\left(a_0 \overrightarrow{N} + \frac{\tau}{\alpha} x_1 \overrightarrow{C} + \frac{\kappa}{\alpha} x_1 \overrightarrow{W}\right),$$

differentiating this last equation, we get

$$\begin{aligned} \overrightarrow{U}' = {} & \frac{1}{\sqrt{a_0^2 + x_1^2}}\left[-\tau x_1 + \frac{a_0' x_1^2}{a_0^2 + x_1^2}\right]\overrightarrow{N} \\ & + \frac{1}{\sqrt{a_0^2 + x_1^2}}\left[\left(\frac{\tau}{\alpha}\right)' x_1 + a_0 \alpha - \kappa\sigma x_1 - \frac{a_0' a_0 \tau x_1}{\alpha\left(a_0^2 + x_1^2\right)}\right]\overrightarrow{C} \\ & + \frac{1}{\sqrt{a_0^2 + x_1^2}}\left[\left(\frac{\kappa}{\alpha}\right)' x_1 + \tau\sigma x_1 - \frac{a_0' a_0 \kappa x_1}{\alpha\left(a_0^2 + x_1^2\right)}\right]\overrightarrow{W}, \end{aligned} \tag{11}$$

on another hand, we havve

$$\overrightarrow{U} \wedge \overrightarrow{T} = \frac{1}{\sqrt{a_0^2 + x_1^2}}\left(x_1 \overrightarrow{N} - \frac{a_0 \tau}{\alpha} \overrightarrow{C} - \frac{a_0 \kappa}{\alpha} \overrightarrow{W}\right), \tag{12}$$

then, if we substitute the equations (11) and (12) in (2) and replace $\overrightarrow{T}'$ by $\kappa \overrightarrow{N}$ we obtain $\rho_n$, $\rho_g$ and $\theta_g$ as follows:

$$\rho_n = \frac{\kappa a_0}{\sqrt{a_0^2 + x_1^2}}, \quad \rho_g = \frac{\kappa x_1}{\sqrt{a_0^2 + x_1^2}}, \quad \theta_g = \tau - \frac{x_1 \alpha\beta\left(-x_2 \kappa + x_3 \tau\right)}{\left(x_2 \tau + x_3 \kappa\right)^2 + \alpha^2 x_1^2}.$$

Consequently, from the above study, one can formulate the following corollaries:

**Corollary 3.1** : $\Psi$ is developable if and only if the curvature $\kappa$ and the torsion $\tau$ of its base curve satisfy the equation $\left(1 - x_3^2 - x_1 x_3 \sigma\right)\tau + x_2 \kappa \left(x_3 + x_1 \sigma\right) = 0$.

**Corollary 3.2** : $\Psi$ is minimal along its base curve $c$ if and only if the curvature $\kappa$ and the torsion $\tau$ of $c$ satisfy the equation $\kappa\left(x_2\tau + x_3\kappa\right)$
$+2\left(-\kappa x_2 + \tau x_3\right)\left[\left(1 - x_3^2 - x_1 x_3 \sigma\right)\tau + x_2\kappa\left(x_3 + x_1\sigma\right)\right] = 0$.

**Corollary 3.3** : $c = c\left(s\right)$ is the striction curve of the ruled surface $\Psi$ if and only if its cuvature $\kappa$ and torsion $\tau$ satisfy the equation $-\kappa\left(x_1 - \sigma x_3\right) + \tau\sigma x_2 = 0$.

**Corollary 3.4** : If $x_1 \neq 0$, then $c = c\left(s\right)$ is an asymptotic line of $\Psi$ if and only if its curvature $\kappa$ and torsion $\tau$ satisfy the equation $x_2\tau + x_3\kappa = 0$.

**Corollary 3.5** : If $x_2\tau + x_3\kappa \neq 0$, then $c = c\left(s\right)$ is a geodesic curve of the ruled surface

$$\left(s, v\right) \mapsto c\left(s\right) + v\left[x_2\overrightarrow{C}\left(s\right) + x_3\overrightarrow{W}\left(s\right)\right].$$

**Corollary 3.6** : $c = c\left(s\right)$ is a principal line of $\Psi$ if an only if its curvature $\kappa$ and torsion $\tau$ satisfy the equation $\tau - \dfrac{x_1\alpha\beta\left(-x_2\kappa + x_3\tau\right)}{\left(x_2\tau + x_3\kappa\right)^2 + \alpha^2 x_1^2} = 0$.

In this following part, we establish the whole of the last results in some special cases by supposing that the ruled surface $\Psi$ is noncylindrical and regular along its base curve.

**Case 1**: $x_1 = 1$ $(x_2 = x_3 = 0)$

The ruled surface becomes

$$\Psi : (s, v) \mapsto c(s) + v\overrightarrow{N}(s). \tag{13}$$

It is a regular surface along its base curve satisfying

$$K(s,0) = -\tau^2, \quad H(s,0) = 0, \quad \rho_n = 0,\ \rho_g = \kappa,\ \theta_g = \tau, \quad \mu = c + \frac{\kappa}{\kappa^2 + \tau^2}\overrightarrow{N},$$

then, we obtain the following corollaries:

**Corollary 3.7** : The ruled surface (13) is minimal along its base curve $c\left(s\right)$ and admits $c\left(s\right)$ as an asymptotic line.

**Corollary 3.8** : For he ruled surface (13) the following properties are equivalents:

1. The ruled surface (13) is developable.
2. The ruled surface (13) admits $c\left(s\right)$ as a principal line.
3. The ruled surface (13) admits $c\left(s\right)$ as a plane curve.

**Case 2**: $x_2 = 1$ $(x_1 = x_3 = 0)$

The ruled surface $\Psi$ will take the folowing form

$$\Psi : (s, v) \in I \times \mathbb{R} \mapsto c(s) + v\overrightarrow{C}(s). \tag{14}$$

For a reason of regularity of (14) along its base curve, we suppose that $\tau \neq 0$.

In this case we have

$$\begin{aligned} K(s,0) &= -\left(\frac{\kappa^2+\tau^2}{\tau}\right)^2, \; H(s,0) = -\frac{\kappa\left(\kappa^2+\tau^2\right)}{2\tau^2}, \; \rho_n = -\kappa, \; \rho_g = 0, \; \theta_g = \tau, \\ \mu &= c - \frac{\tau\kappa^2\sqrt{\kappa^2+\tau^2}}{\left(\kappa^2+\tau^2\right)^3 + \kappa^4\left[\left(\frac{\tau}{\kappa}\right)'\right]^2}\left(\frac{\tau}{\kappa}\right)'\overrightarrow{C}, \end{aligned}$$

then we get the following corollaries:

**Corollary 3.9** : The ruled surface (14) admits

1. $c(s)$ as a geodesic curve.
2. $c(s)$ as striction curve if and only if $c(s)$ is a general helix.

**Remark 3.10** : The ruled surface (14) is neither developable nor minimal along its base curve.

**Case 3**: $x_3 = 1$ $(x_1 = x_2 = 0)$

The ruled surface $\Psi$ becomes

$$\Psi : (s, v) \in I \times \mathbb{R} \mapsto c(s) + v\overrightarrow{W}(s). \tag{15}$$

It is a regular surface along its base curve satisfying:

$$K(s,0) = 0, \quad H(s,0) = \frac{\kappa^2+\tau^2}{2\kappa}, \quad \rho_n = -\kappa, \; \rho_g = 0, \; \theta_g = \tau,$$

one suppose that $c(s)$ is not a general helix is non-cylindrical, then its striction curve is

$$\mu = c - \frac{\sqrt{1+\left(\frac{\tau}{\kappa}\right)^2}}{\left(\frac{\tau}{\kappa}\right)'}\overrightarrow{W},$$

hence, we get the following corollaries:

**Corollary 3.11** : The ruled surface (15) is

1. developable.
2. admits $c(s)$ as a geodesic curve.
3. admits $c(s)$ as a principal line if and only $c(s)$ is a plane curve.

## 4. Ruled Surfaces with Alternative Moving Frame Generated by Some Special General and Slant Helices

In this section and with illustrations, we presente ruled surfaces with alternative moving frame which are generated by some special Genaral and Slant helices [8, 9].

**4.1 Ruled surfaces generated by General helices**

In the following, ruled surfaces generated by some special general helices [8] such as circular helix, spherical helix and another case of general helix are presented.

**Theorem 4.1 [8]** : The position vector $c$ of general helix whose tangent vector makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:

$$c(s) = \frac{1}{\sqrt{1+m^2}} \int \left( \cos\left( \sqrt{1+m^2} \int \kappa(s)\, ds \right), \sin\left( \sqrt{1+m^2} \int \kappa(s)\, ds \right), m \right) ds, \tag{16}$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos(\phi)$, $\phi$ is the angle between the tangent vector of the curve $c$ and a fixed direction (axis general helix).

Then, alternative moving frame vectors $\overrightarrow{N}$, $\overrightarrow{C}$ and $\overrightarrow{W}$ of the general helix (16) are respectively as follows:

$$\overrightarrow{N} = \begin{pmatrix} -\sin\left(\sqrt{1+m^2}\int \kappa(s)\, ds\right) \\ \cos\left(\sqrt{1+m^2}\int \kappa(s)\, ds\right) \\ 0 \end{pmatrix}, \quad \overrightarrow{C} = \begin{pmatrix} -\cos\left(\sqrt{1+m^2}\int \kappa(s)\, ds\right) \\ -\sin\left(\sqrt{1+m^2}\int \kappa(s)\, ds\right) \\ 0 \end{pmatrix}, \quad \overrightarrow{W} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

then, position vector of the ruled surface $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ is

$$\begin{cases} \Psi_1 = \dfrac{1}{\sqrt{1+m^2}} \int \cos(\Theta)\, ds - v\left[x_1 \sin(\Theta) + x_2 \cos(\Theta)\right], \\ \Psi_2 = \dfrac{1}{\sqrt{1+m^2}} \int \sin(\Theta)\, ds + v\left[x_1 \cos(\Theta) - x_2 \sin(\Theta)\right], \\ \Psi_3 = \dfrac{ms}{\sqrt{1+m^2}} + v x_3, \end{cases}$$

where $\Theta = \sqrt{1+m^2}\int \kappa(s)\,ds$.

**Case 1:** Considering general circular helix defined by intrinsic equations $\kappa(s) = \kappa$, $\tau(s) = m\kappa$.

The ruled surface $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ generated by general circular helix is:

$$\begin{cases} \Psi_1 = \dfrac{\sin(\Theta)}{\kappa(1+m^2)} - v\left[x_1 \sin(\Theta) + x_2 \cos(\Theta)\right], \\ \Psi_2 = -\dfrac{\cos(\Theta)}{\kappa(1+m^2)} + v\left[x_1 \cos(\Theta) - x_2 \sin(\Theta)\right], \\ \Psi_3 = \dfrac{ms}{\sqrt{1+m^2}} + vx_3, \end{cases}$$

where $\Theta = \sqrt{1+m^2}\kappa s$

**Case 2** : Considering general circular defined by intrinsic equations $\kappa(s) = \dfrac{a}{s}$, $\tau(s) = \dfrac{ma}{s}$.

Ruled surface $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ generated by this kind of general helix is:

$$\begin{cases} \Psi_1 = \dfrac{1}{\sqrt{1+m^2}}\left[\frac{s}{1+b^2}\cos(\Theta) + \frac{as}{1+b^2}\sin(\Theta)\right] - v\left[x_1 \sin(\Theta) + x_2 \cos(\Theta)\right], \\ \Psi_2 = \dfrac{1}{\sqrt{1+m^2}}\left[\frac{s}{1+b^2}\sin(\Theta) - \frac{as}{1+b^2}\cos(\Theta)\right] + v\left[x_1 \cos(\Theta) - x_2 \sin(\Theta)\right], \\ \Psi_3 = \dfrac{ms}{\sqrt{1+m^2}} + vx_3, \end{cases}$$

where $b = a\sqrt{1+m^2}$, $\Theta = b\log(s)$.

**Case 3:** Considering general spherical helix defined by intrinsic equations $\kappa(s) = \dfrac{a}{\sqrt{1-m^2s^2}}$, $\tau(s) = \dfrac{ma}{\sqrt{1-m^2s^2}}$.

Then, ruled surface $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ generated by general spherical helix is

$$\begin{cases} \Psi_1 = \dfrac{1}{a^2(1+m^2) - m^2}\left[-nms\cos(\Theta) + a\sqrt{1-m^2s^2}\sin(\Theta)\right] - v\left[x_1 \sin(\Theta) + x_2 \cos(\Theta)\right], \\ \Psi_2 = -\dfrac{1}{a^2(1+m^2) - m^2}\left[nms\sin(\Theta) + a\sqrt{1-m^2s^2}\cos(\Theta)\right] + v\left[x_1 \cos(\Theta) - x_2 \sin(\Theta)\right], \\ \Psi_3 = \dfrac{ms}{\sqrt{1+m^2}} + vx_3, \end{cases}$$

where $\Theta = \frac{a}{n}\arcsin(ms)$.

### 4.2 Ruled surfaces generated by slant helices

**Theorem 4.2 [9]** : The position vector $c$ of slant helix whose principal normal vector makes a constant angle with a fixed straight line in the space, is expressed in the natural representation form as follows:

$$\begin{cases} c_1(s) = \frac{n}{m}\int\left[\int\kappa(s)\cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right]ds\right]ds, \\ c_2(s) = \frac{n}{m}\int\left[\int\kappa(s)\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right]ds\right]ds, \\ c_3(s) = n\int\left[\int\kappa(s)\,ds\right]ds, \end{cases} \tag{17}$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$, $n = \cos(\phi)$, $\phi$ is the angle between the principal normal vector of the curve $c$ and a fixed direction (axis of slant helix).

Thus, from the last theorem one can get the alternative moving frame vectors $\overrightarrow{N}$, $\overrightarrow{C}$ and $\overrightarrow{W}$ respectively as follows:

$$\overrightarrow{N} = \begin{pmatrix} \frac{n}{m}\cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right] \\ \frac{n}{m}\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right] \\ n \end{pmatrix}, \quad \overrightarrow{C} = \begin{pmatrix} -\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right] \\ \cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right] \\ 0 \end{pmatrix} \tag{18}$$

$$\overrightarrow{W} = \begin{pmatrix} -n\cos\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right] \\ -n\sin\left[\frac{1}{n}\arcsin\left(m\int\kappa(s)\,ds\right)\right] \\ \frac{n}{m} \end{pmatrix}, \tag{19}$$

hence, the ruled surface $\Psi = (\Psi_1, \Psi_2, \Psi_3)$ generated by slant helix (17) and defined by its alternative moving frame vectors (18) and (19) is

$$\begin{cases} \Psi_1 = \frac{n}{m}\int\left[\int\kappa(s)\cos[\Phi]\,ds\right]ds + v\left[\frac{n}{m}x_1\cos[\Phi] - x_2\sin[\Phi] - nx_3\cos[\Phi]\right], \\ \Psi_2 = \frac{n}{m}\int\left[\int\kappa(s)\sin[\Phi]\,ds\right]ds + v\left[\frac{n}{m}x_1\sin[\Phi] + x_2\cos[\Phi] - nx_3\sin[\Phi]\right], \\ \Psi_3 = \frac{n}{m}\left[\int\Theta ds + v(mx_1 + x_3)\right], \end{cases}$$

where $\Theta = m\int\kappa(s)\,ds$ and $\Phi = \frac{1}{n}\arcsin(\Theta)$.

In the following, ruled surfaces generated by some special slant helices [9] such as Salkowski curve, anti-Salkowski curve and another case of slant helix are presented.

**Case 1:** Considering Salkowski curve as a special case of slant helix which is defined by the intrinsic equations $\kappa(s) = 1$, $\tau(s) = \dfrac{ms}{\sqrt{1-m^2s^2}}$.

The components $\Psi_1$, $\Psi_2$ and $\Psi_3$ of ruled surface generated by Salkowski curve and defined by its alternative moving frame vectors are:

$$\begin{cases} \Psi_1 = \frac{n}{4m}\left[\frac{n-1}{2n+1}\cos[(2n+1)t] + \frac{n+1}{2n-1}\cos[(2n-1)t] - 2\cos[t]\right] \\ \qquad + v\left[\frac{n}{m}x_1\cos[t] - x_2\sin[t] - nx_3\cos[t]\right] \\ \Psi_2 = \frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t] - \frac{n+1}{2n-1}\sin[(2n-1)t] - 2\sin[t]\right] \\ \qquad + v\left[\frac{n}{m}x_1\sin[t] + x_2\cos[t] - nx_3\sin[t]\right] \\ \Psi_3 = -\frac{n}{m}\left[\frac{1}{4m}\cos[2nt] - v(mx_1 + x_3)\right] \end{cases}$$

where $t = \dfrac{1}{n}\arcsin(ms)$.

**Case 2:** Anti-Salkowski curve is defined by the intrinsic equations $\kappa(s) = \dfrac{ms}{\sqrt{1-m^2s^2}}$, $\tau(s) = 1$.

The components $\Psi_1$, $\Psi_2$ and $\Psi_3$ of ruled surface generated by anti-salkowski curve and defined by its alternative moving frame vectors are:

$$\begin{cases} \Psi_1 &= \frac{n}{4m}\left[\frac{n-1}{2n+1}\sin[(2n+1)t] + \frac{n+1}{2n-1}\sin[(2n-1)t] - 2n\sin[t]\right] \\ & +v\left[\frac{n}{m}x_1\cos[t] - x_2\sin[t] - nx_3\cos[t]\right] \\ \Psi_2 &= \frac{n}{4m}\left[-\frac{n-1}{2n+1}\cos[(2n+1)t] + \frac{n+1}{2n-1}\cos[(2n-1)t] + 2n\cos[t]\right] \\ & +v\left[\frac{n}{m}x_1\sin[t] + x_2\cos[t] - nx_3\sin[t]\right] \\ \Psi_3 &= \frac{n}{m}\left[\frac{1}{4m}(2nt - \sin[2nt]) + v(mx_1 + x_3)\right] \end{cases}$$

where $t = \frac{1}{n}\arcsin(m\theta)$ and $\theta = \dfrac{\sqrt{1-m^2s^2}}{m}$.

**Case 3:** Considering slant helix which is defined by the following intrinsic equations

$$\kappa(s) = \frac{\mu}{m}\cos[\mu s], \tau(s) = \frac{\mu}{m}\sin[\mu s]. \tag{20}$$

The components $\Psi_1$, $\Psi_2$ and $\Psi_3$ dof ruled surface generated by the slant helix (20) and defined by its alternative moving frame vectors are:

$$\begin{cases} \Psi_1 &= -\frac{m^2}{n\mu}\left[\left(1+n^2\right)\cos[\mu s]\cos\left[\frac{\mu s}{n}\right] + 2n\sin[\mu s]\sin\left[\frac{\mu s}{n}\right]\right] \\ & +v\left[\frac{n}{m}x_1\cos\left[\frac{\mu s}{n}\right] - x_2\sin\left[\frac{\mu s}{n}\right] - nx_3\cos\left[\frac{\mu s}{n}\right]\right] \\ \Psi_2 &= -\frac{m^2}{n\mu}\left[\left(1+n^2\right)\cos[\mu s]\sin\left[\frac{\mu s}{n}\right] - 2n\sin[\mu s]\cos\left[\frac{\mu s}{n}\right]\right] \\ & +v\left[\frac{n}{m}x_1\sin\left[\frac{\mu s}{n}\right] + x_2\cos\left[\frac{\mu s}{n}\right] - nx_3\sin\left[\frac{\mu s}{n}\right]\right] \\ \Psi_3 &= \frac{n}{m}\left[-\frac{1}{\mu}\cos[\mu s] + v(mx_1 + x_3)\right]. \end{cases}$$

In what follows, some illustrations of ruled surfaces generated by the last cases of general and slant helices and which are defined by their alternative moving frame vectors are presented:

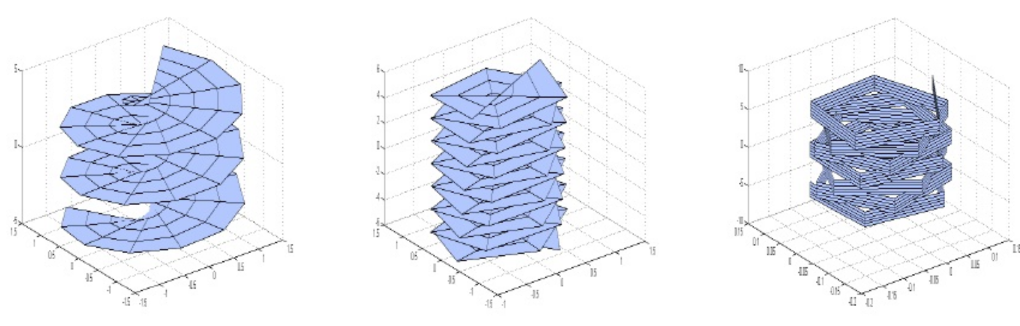

**Figure** 1 : Some ruled surfaces generated by circular general helix.

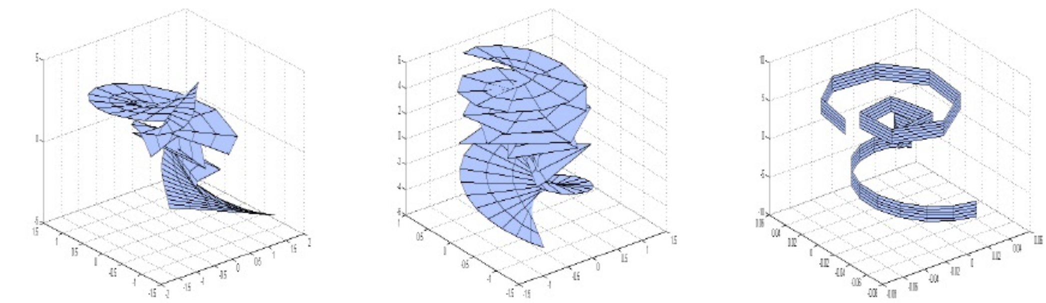

**Figure** 2 : Some ruled surfaces generated by the general helix with the intrinsic equations $\kappa(s) = \frac{a}{s},\ \tau(s) = \frac{ma}{s}$.

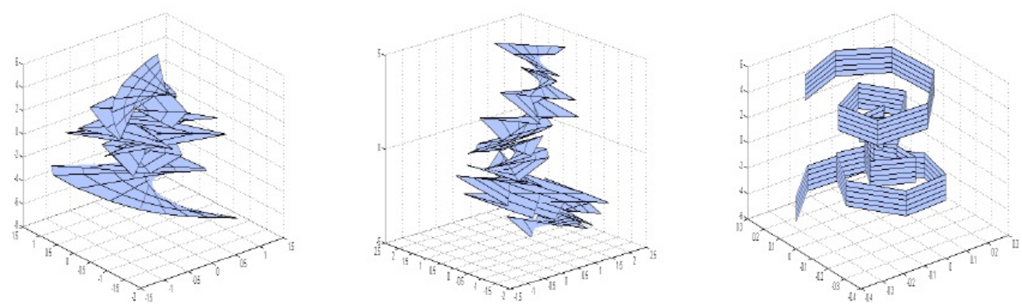

**Figure** 3 : Some ruled surfaces generated by spherical general helix.

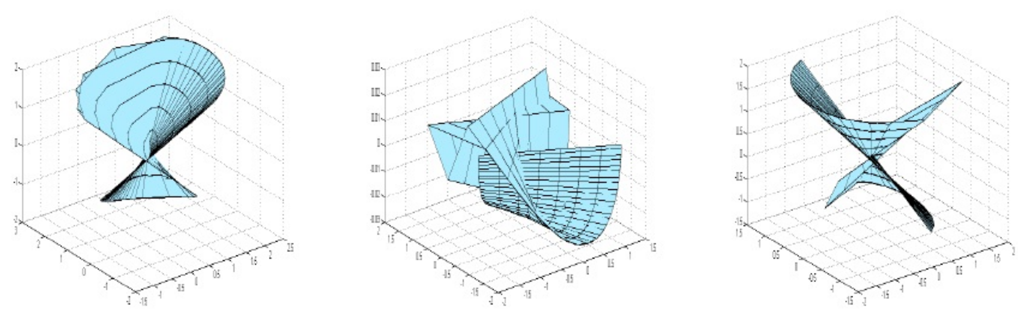

**Figure** 4 : Some ruled surfaces generated by Salkowski curve.

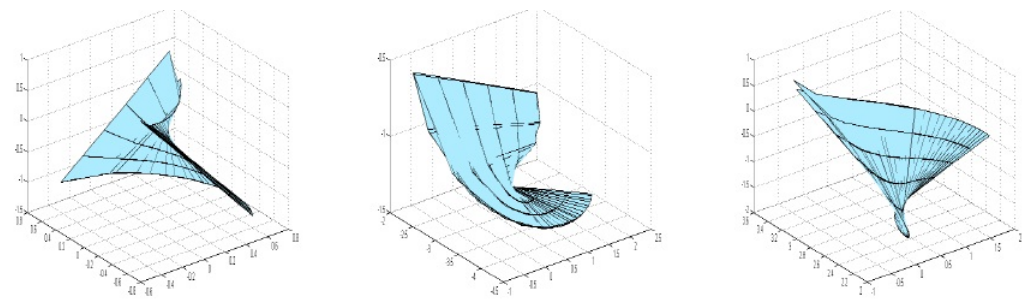

**Figure** 5 : Some ruled surfaces generated by anti-Salkowski curve.

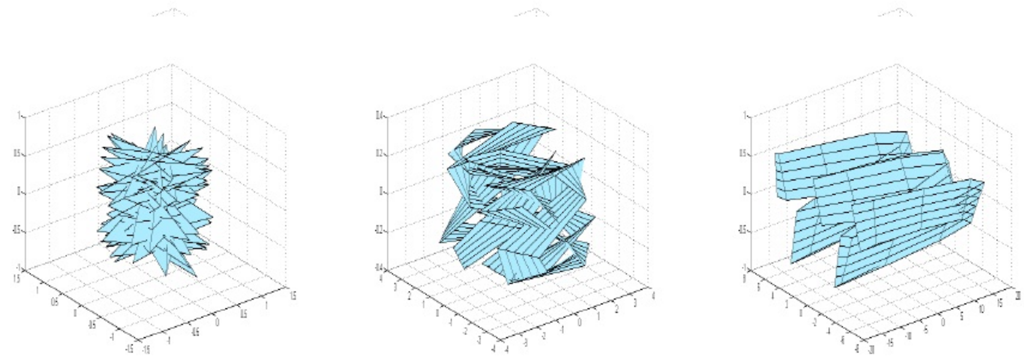

**Figure 6** : Some ruled surfaces generated by the slant helix with the intrinsic equations

$$\kappa(s) = \frac{\mu}{m}\cos[\mu s],.\ \tau(s) = \frac{\mu}{m}\sin[\mu s].$$

**Fig. 1:** $L : (m = \kappa = 1,\ x_1 = 1,\ x_2 = x_3 = 0), M : (m = 3,\ \kappa = 1,\ x_2 = 1,\ x_1 = x_3 = 0),$ $R : \left(m = 2,\ \kappa = \frac{3}{2},\ x_3 = 1,\ x_1 = x_2 = 0\right).$

**Fig. 2:** $L : (a = 2,\ m = 1,\ x_1 = 1,\ x_2 = x_3 = 0), M : (a = 2,\ m = 3,\ x_2 = 1,\ x_1 = x_3 = 0),$ $R : (a = m = 3,\ x_3 = 1,\ x_1 = x_2 = 0).$

**Fig. 3:** $L : (a = 3,\ m = 4,\ x_1 = 1,\ x_2 = x_3 = 0), M : (a = 2,\ m = 1,\ x_2 = 1,\ x_1 = x_3 = 0),$ $R : (a = 7,\ m = 5,\ x_3 = 1,\ x_1 = x_2 = 0).$

**Fig. 4:** $L : (m = 1,\ x_1 = 1,\ x_2 = x_3 = 0),\ M : (m = 3,\ x_2 = 1,\ x_1 = x_3 = 0),\ R :$ $(m = 4,\ x_3 = 1,\ x_1 = x_2 = 0).$

**Fig. 5:** $L : (m = 2,\ x_1 = 1,\ x_2 = x_3 = 0),\ M : \left(m = \frac{1}{2},\ x_2 = 1,\ x_1 = x_3 = 0\right),\ R :$ $\left(m = \frac{2}{3},\ x_3 = 1,\ x_1 = x_2 = 0\right).$

**Fig. 6:** $L : \left(m = \frac{1}{2},\ \mu = 4,\ x_1 = 1,\ x_2 = x_3 = 0\right), M : \left(m = \frac{3}{2},\ \mu = 2, x_2 = 1, x_1 = x_3 = 0\right),$ $R : (m = 3,\ \mu = 1,\ x_3 = 1,\ x_1 = x_2 = 0).$

**Remark 4.3** : The symbols (L, M and R) mean (Left, Middle and Right) in the graphs, respectively.

## References


[1] Şenyurt S., Çalişkan A., Principal normal vectors belonging to striction curves of Frenet and Bertrandian Frenet ruled surfaces, American Institute of Physics (2017).

[2] Soliman M. A. , Abd-Ellah H. N., Hassa S. A.n, Saleh S. Q., Darboux ruled surfaces with pointwise 1-type Gauss map, Sohag J. Sci. 2(1) (2017), 1-8.

[3] Uzunõglu,B., Gök I., Yayli Y., A new approach on curves of constant precession, Applied Mathematics and Computation, 275 (2016), 317-323.

[4] Unluturk Y., Cimdiker M., Ekici C., Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3- space, CMMA 1(1) (2016), 26-43.

[5] Sentürk G. Y., Yüce S., Characteristic properties of the ruled surface with Darboux frame in $E^3$, Kuwait J. Sci., 42(2) (2015), 14-33.

[6] Ali H. T., Abdel Aziz H. S., Sorour A. H., Ruled surfaces generated by some special curves in Euclidean 3-Space, J. Egyptian. Math. Soc., 21 (2013), 285-294.

[7] Kiziltug S., çakmak A., Developable Ruled Surfaces with Darboux Frame in Minkowski 3-Space, Life Science Journal, 10(4) (2013).

[8] Ali A. T., Position vectors of general helices in Euclidean space, Bulletin of Mathematical Analysis and Applications, (2011).

[9] Ali A. T., Position vectors of slant helices in Euclidean space $E^3$, arXiv:0907.0750v1 [math.DG], (2009).

[10] Izumiya S., Takeuchi N., New special curves and developable surfaces, Turk. J. Math., 28 (2004), 531-537.

[11] Do-carmo P., Differential geometry of curves and surfaces, IMPA, (1976), 511.

[12] Struik D. J., Lectures on Classical Differential Geometry, Addison-Wesley Publishing Company, Inc., (1961).