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Ruled Surface Generated by a Curve Lying on a Regular Surface and its Characterizations

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Abstract. In this paper, we construct and study special ruled surfaces, whose rulings are linear combinations of Darboux frame vectors of its base curve relative to an arbitrary regular surface in euclidean 3-space. We give the relationship between both Darboux frames of the common curve relative to the two surfaces, and investigate different properties of the constructed ruled surface. Moreover, we presente three examples with illustrations in the real euclidean 3-space.

Key Words: ruled surface, Darboux frame, striction curve, Gaussian curvature, mean curvature, Euclidean 3-space

MSC 2020: 53A05

1 Introduction

In differential geometry, we think of a ruled surface as a geometric set of lines. Surface theory is an important workdomain in differential geometry studies. It has been the point of interest for many researchers [4, 6, 15, 17].

A ruled surface is one of the special surfaces. It is defined by the moving of a straight line (ruling) along a curve (base curve). This fascinating special surface is of great interest to many applications and has contribution in several areas, such as mathematical physics, kinematics and Computer Aided Geometric Design (CAGD) [7, 11].

Nowadays, a good deal of research on ruled surface theory has been conducted about ruled surfaces in Euclidean and Minkowski space [9, 10, 18].

In [3], the authors studied the ruled surface whose rulings are linear combinations of Frenet frame vectors of its base curve. They gave its position vector in the case of the base curve as general helix [2] and slant helix [12], respectively. Furthermore, in [14] the authors were interested in the study of ruled surface with alternative moving frame of its base curve. They investigated its most important properties and gave characterizations. Finally, in [1, 8, 13, 19], we can find a study of ruled surface according to Bishop frame vectors of its base curve.

On the other hand, in [5, 16], the authors expressed ruled surfaces by means of the Darboux frame vectors of its base curve and they studied some properties and integral invariants of the same ruled surface.

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In this paper, we consider Darboux frame vectors of a unit speed curve $c(s)$ lying on a regular surface φ and we construct the ruled surface Ψ which takes $c(s)$ as base curve and whose rulings are constant linear combinations of Darboux frame vectors of $c(s)$ relative to the initial surface φ. We propose to make a comparative study of the two surfaces, the regular surface φ and the ruled surface Ψ. More exactly, we give the relation between the two Darboux frames of the common curve $c(s)$ relatively to φ and Ψ, respectively. Futhermore, we investigate the main properties of the constructed ruled surface and characterize it along its base curve. In the last section, we present examples with illustrations.

2 Preliminaries

In this section, we will present some basic concepts related to regular surfaces, the Darboux frame of a curve lying on a regular surface and the notion of ruled surface.

Let E^3 be a 3-dimensional euclidean space equipped with the metric given by $\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a rectangular coordinate system of E^3.

To a unit speed regular curve $c = c(s)$ that lies on a regular surface, there exists the Darboux frame denoted $\{\vec{T}(s), \vec{g}(s), \vec{h}(s)\}$, where $\vec{T}(s)$ is the unit tangent vector of the curve $c = c(s)$, $\vec{h}(s)$ is the unit normal vector of the surface along $c = c(s)$ and $\vec{g}(s)$ is the unit vector defined by the cross product of $\vec{h}(s)$ et $\vec{T}(s)$, i.e., $\vec{g}(s) = \vec{h}(s) \wedge \vec{T}(s)$.

The derivative formulae of the Darboux frame are given as follows:

$$\begin{bmatrix} \vec{T}'(s) \\ \vec{g}'(s) \\ \vec{h}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \rho_g(s) & \rho_n(s) \\ -\rho_g(s) & 0 & \theta_g(s) \\ -\rho_n(s) & -\theta_g(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{g}(s) \\ \vec{h}(s) \end{bmatrix}, \tag{1}$$

where $\rho_n(s)$ is the normal curvature, $\rho_g(s)$ is the geodesic curvature and $\theta_g(s)$ is the geodesic torsion of $c(s)$ on the regular surface, and they can be calculated as follows

$$\rho_n = \left\langle \vec{h}, \vec{T}' \right\rangle, \quad \rho_g = \left\langle \vec{g}, \vec{T}' \right\rangle, \quad \theta_g = -\left\langle \vec{g}, \vec{h}' \right\rangle.$$

definition 1 ([6, 15])**.** The curve $c(s)$ lying on a regular surface

- is an asymptotic line if its normal curvature ρ_n vanishes.
- is a geodesic curve if its geodesic curvature ρ_g vanishes.
- is a principal line if its geodesic torsion θ_g vanishes.

The standard unit normal vector on a regular surface $\varphi = \varphi(u, t)$ is identified by:

$$\frac{\varphi_u \wedge \varphi_t}{\|\varphi_u \wedge \varphi_t\|}, \quad \text{where } \varphi_u = \frac{\partial \varphi(u,t)}{\partial u} \text{ and } \varphi_t = \frac{\partial \varphi(u,t)}{\partial t}.$$

The first fundamental form I and the second fundamental form II of a regular surface $\varphi = \varphi(u, t)$ are defined on its tangent plane, which is generated by $\{\varphi_u, \varphi_t\}$. They are given respectively by

$$I(\varphi_u du + \varphi_t dt) = E\, du^2 + 2F\, du\, dt + G\, dt^2, \quad II(\varphi_u du + \varphi_t dt) = e\, du^2 + 2f\, du\, dt + g\, dt^2$$

where

$$E = \|\varphi_u\|^2, \quad F = \langle \varphi_u, \varphi_t \rangle, \quad G = \|\varphi_t\|^2,$$
$$e = \left\langle \varphi_{uu}, \frac{\varphi_u \wedge \varphi_t}{\|\varphi_u \wedge \varphi_t\|} \right\rangle, \quad f = \left\langle \varphi_{ut}, \frac{\varphi_u \wedge \varphi_t}{\|\varphi_u \wedge \varphi_t\|} \right\rangle, \quad g = \left\langle \varphi_{tt}, \frac{\varphi_u \wedge \varphi_t}{\|\varphi_u \wedge \varphi_t\|} \right\rangle.$$

The Gaussian curvature K and the mean curvature H of the regular surface $\varphi = \varphi(u,t)$ are defined by

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)},$$

respectively.

definition 2 ([6, 15])**.** A regular surface is minimal if its mean curvature vanishes identically.

On the other hand, a ruled surface in E^3 is generated by a one-parameter family of straight lines and has the parametric representation

$$\begin{array}{rccl} \Psi\colon & I \times \mathbb{R} & \longrightarrow & E^3 \\ & (s,v) & \longmapsto & c(s) + v\vec{X}(s), \end{array}$$

where I is an open interval of the real line $\mathbb{R}$. $c = c(s)$ is called the base curve of the ruled surface and $\vec{X}(s)$ are the unit vectors representing the direction of straight lines (rulings).

The ruled surface is said to be a non-cylindrical ruled surface provided that $\|\vec{X}'(s)\| \neq 0$.

definition 3 ([6, 15])**.** A ruled surface is developable if at any point we have $\langle c'(s) \wedge \vec{X}(s), \vec{X}'(s)\rangle = 0$.

For ruled surface, we have

$$f(s,v) = \frac{\langle c'(s) \wedge \vec{X}(s), \vec{X}'(s)\rangle}{\|\Psi_s \wedge \Psi_v\|}, \quad g(s,v) = 0,$$

from where

$$K = -\frac{f^2}{EG - F^2}, \quad H = \frac{Ge - 2Ff}{2(EG - F^2)}.$$

Hence, we have the following theorem:

theorem 1 ([6, 15])**.** *A ruled surface is developable if and only if its Gaussian curvature vanishes identically.*

definition 4 ([6, 15])**.** The striction curve of a non-cylindrical ruled surface is the unique curve $\beta = \beta(s)$ lying on the ruled surface and satisfying $\langle \beta'(s), \vec{X}'(s)\rangle = 0$.

It is parametrized by

$$\beta(s) = c(s) - \frac{\langle c'(s), \vec{X}'(s)\rangle}{\|\vec{X}'(s)\|^2}\vec{X}(s).$$

3 Ruled surface generated by a curve lying on a regular surface and their characterizations

In our study, we consider a unit speed curve $c = c(s)$ lying on a regular surface $\varphi = \varphi(u,t)$ and we construct the ruled surface $\Psi = \Psi(s,v)$ taking $c(s)$ as a base curve and for direction vectors of rulings, a linear combination of Darboux frame vectors of $c = c(s)$ with respect to the initial surface $\varphi = \varphi(u,t)$.

Let $c = c(s)\colon I \subset \mathbb{R} \to E^3$ be a unit speed regular curve lying on a regular surface $\varphi = \varphi(u,t)$, i.e.

$$c(s) = \varphi(u(s), t(s)), \quad \|c'(s)\| = 1, \quad \forall s \in I.$$

We denote by $\{\vec{T}(s), \vec{g}(s), \vec{h}(s)\}$ and by $\rho_n(s)$, $\rho_g(s)$ and $\theta_g(s)$ the Darboux frame, the normal curvature, the geodesic curvature and the geodesic torsion of $c = c(s)$ on the surface φ, respectively.

Considering the ruled surface $\Psi = \Psi(s,v)$ defined in E^3 by

$$\Psi\colon (s,v) \mapsto c(s) + v\vec{X}(s), \quad \vec{X}'(s) \neq 0, \tag{2}$$

where

$$\vec{X}(s) = a_1\vec{T}(s) + a_2\vec{g}(s) + a_3\vec{h}(s)$$

is a unit vector with fixed components, i.e., $a_1^2 + a_2^2 + a_3^2 = 1$.

Note that in all the rest of this paper, we are indexing invariants of ruled surface $\Psi = \Psi(s,v)$ by $^{(r)}$.

Differentiating (2) with respect to s and v, and using Darboux formulae (1), we get

$$\begin{cases} \Psi_s &= [1 - v(a_2\rho_g + a_3\rho_n)]\vec{T} + v(a_1\rho_g - a_3\theta_g)\vec{g} + v(a_1\rho_n + a_2\theta_g)\vec{h}, \\ \Psi_v &= a_1\vec{T} + a_2\vec{g} + a_3\vec{h}. \end{cases} \tag{3}$$

Then, the cross product gives

$$\begin{aligned} \Psi_s \wedge \Psi_v = \; & v\{a_3(a_1\rho_g - a_3\theta_g) - a_2(a_1\rho_n + a_2\theta_g)\}\vec{T} \\ & + \{-a_3 + v[a_3(a_2\rho_g + a_3\rho_n) + a_1(a_1\rho_n + a_2\theta_g)]\}\vec{g} \\ & + \{a_2 - v[a_2(a_2\rho_g + a_3\rho_n) - a_1(a_1\rho_g - a_3\theta_g)]\}\vec{h}. \end{aligned}$$

From this last equation, we can see that regularity condition of ruled surface $\Psi = \Psi(s,v)$ on its base curve $c = c(s)$ is $a_3^2 + a_2^2 \neq 0$. In this case, the unit normal on $\Psi = \Psi(s,v)$ along its base curve $c = c(s)$ is given by

$$\frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|}(s,0) = \frac{1}{\sqrt{a_2^2 + a_3^2}}(-a_3\vec{g}(s) + a_2\vec{h}(s)). \tag{4}$$

Thus, the unit normal denoted by $\vec{h}^{(r)}$ of the ruled surface Ψ and the unit normal $\vec{h}$ of the surface φ are related on their common curve $c = c(s)$ as

$$\vec{h}^{(r)} = \frac{1}{\sqrt{a_2^2 + a_3^2}}(a_3\vec{T} \wedge \vec{h} + a_2\vec{h}). \tag{5}$$

From (4) and (5), the normal curvature $\rho_n^{(r)}$, the geodesic curvature $\rho_g^{(r)}$ and the geodesic torsion $\theta_g^{(r)}$ of the curve $c = c(s)$ relative to ruled surface Ψ, are obtained as follows

$$\rho_n^{(r)} = \frac{-a_3\rho_g + a_2\rho_n}{\sqrt{a_2^2 + a_3^2}}, \quad \rho_g^{(r)} = \frac{a_3\rho_n + a_2\rho_g}{\sqrt{a_2^2 + a_3^2}}, \quad \theta_g^{(r)} = \theta_g.$$

corollary 1. *$c = c(s)$ is a principal line for Ψ if and only if $c = c(s)$ is a principal line for φ.*

Applying the norms of equations (3), we obtain the components of the first fundamental form of Ψ as follows

$$\begin{cases} E^{(r)} = 1 - 2v(a_2\rho_g + a_3\rho_n) + v^2[(a_2\rho_g + a_3\rho_n)^2 + (a_1\rho_g - a_3\theta_g)^2 + (a_1\rho_n + a_2\theta_g)^2], \\ F^{(r)} = a_1, \\ G^{(r)} = 1. \end{cases} \tag{6}$$

Differentiating both equations in (3) with respect to s and v, and using Darboux formulae (1), we get

$$\begin{cases} \Psi_{ss} &= -v\{(a_2\rho_g' + a_3\rho_n') + \rho_g(a_1\rho_g - a_3\theta_g) + \rho_n(a_1\rho_n + a_2\theta_g)\}\vec{T} \\ &\quad +\{\rho_g + v[-\rho_g(a_2\rho_g + a_3\rho_n) + (a_1\rho_g' - a_3\theta_g') + \theta_g(a_1\rho_n + a_2\theta_g)]\}\vec{g} \\ &\quad +\{\rho_n + v[-\rho_n(a_2\rho_g + a_3\rho_n) + (a_1\rho_n' + a_2\theta_g') - \theta_g(a_1\rho_g - a_3\theta_g)]\}\vec{h}, \\ \Psi_{sv} &= -(a_2\rho_g + a_3\rho_n)\vec{T}(s) + (a_1\rho_g - a_3\theta_g)\vec{g}(s) + (a_1\rho_n + a_2\theta_g)\vec{h}(s), \\ \Psi_{vv} &= 0. \end{cases} \tag{7}$$

Then, from (4) and (7), the components of the second fundamental form of ruled surface Ψ along its base curve are obtained as follows

$$\begin{cases} e^{(r)}(s,0) = \dfrac{-a_3\rho_g(s) + a_2\rho_n(s)}{\sqrt{a_2^2 + a_3^2}}, \\ f^{(r)}(s,0) = \dfrac{-a_3(a_1\rho_g(s) - a_3\theta_g(s)) + a_2(a_1\rho_n(s) + a_2\theta_g(s))}{\sqrt{a_2^2 + a_3^2}}, \\ g^{(r)}(s,0) = 0. \end{cases} \tag{8}$$

Hence, from (6) and (8), the Gaussian curvature $K^{(r)}$ and the mean curvature $H^{(r)}$ of ruled surface Ψ along its base curve are computed as

$$K^{(r)}(s,0) = -\frac{[-(a_3a_1)\rho_g(s) + (a_2a_1)\rho_n(s) + (a_2^2 + a_3^2)\theta_g(s)]^2}{(a_2^2 + a_3^2)^2},$$

$$H^{(r)}(s,0) = \frac{-a_3(1 - 2a_1^2)\rho_g(s) + a_2(1 - 2a_1^2)\rho_n(s) - 2a_1(a_3^2 + a_2^2)\theta_g(s)}{2(a_2^2 + a_3^2)^{3/2}}.$$

On the other hand, under the condition that ruled surface Ψ is non-cylindrical, i.e., $(a_2\rho_g + a_3\rho_n)^2 + (a_1\rho_g - a_3\theta_g)^2 + (a_1\rho_n + a_2\theta_g)^2 \neq 0$, its striction curve which we denote by $\beta^{(r)} = \beta^{(r)}(s)$ is given by

$$\beta^{(r)} = c + \frac{a_2\rho_g + a_3\rho_n}{(a_2\rho_g + a_3\rho_n)^2 + (a_1\rho_g - a_3\theta_g)^2 + (a_1\rho_n + a_2\theta_g)^2}(a_1\vec{T} + a_2\vec{g} + a_3\vec{h}).$$

corollary 2. *$c = c(s)$ is the striction curve of Ψ if and only if it is a geodesic curve for Ψ.*

In the following part, we will investigate, along the common curve $c = c(s)$, the relation between the unit normals $\vec{h}^{(r)}$ and $\vec{h}$, the relation between $\rho_n^{(r)}$, $\rho_g^{(r)}$ and ρ_n, ρ_g, and compute the Gaussian curvature $K^{(r)}$, the mean curvature $H^{(r)}$ and the striction curve $\beta^{(r)} = \beta^{(r)}(s)$ of the constructed ruled surface Ψ in some special cases, under the regularity condition, i.e., $a_2^2 + a_3^2 \neq 0$.

Case 1: For $a_2 = 1$ $(a_1 = a_3 = 0)$, we have

$$\vec{h}^{(r)} = \vec{h},\ \rho_n^{(r)} = \rho_n,\ \rho_g^{(r)} = \rho_g,\ K^{(r)}(s,0) = -\theta_g^2,\ H^{(r)}(s,0) = \frac{\rho_n}{2},\ \beta^{(r)} = c + \frac{\rho_g}{\rho_g^2 + \theta_g^2}\vec{g}.$$

corollary 3. *For $a_2 = 1$ $(a_1 = a_3 = 0)$, the curve $c(s)$ is a geodesic curve (resp. an asymptotic line) on (2) if and only if $c(s)$ is a geodesic curve (resp. an asymptotic line) on φ.*

corollary 4. *For $a_2 = 1$ $(a_1 = a_3 = 0)$, the following characterizations are satisfied:*

1. *The ruled surface* (2) *is developable if and only if* $c(s)$ *is a principal line on* φ.
2. *The ruled surface* (2) *is minimal along its base curve* $c(s)$ *if and only if* $c(s)$ *is an asymptotic line on* φ.
3. $c(s)$ *is the striction curve of* (2) *if and only if* $c(s)$ *is a geodesic curve on* φ.

Case 2: For $a_3 = 1$ ($a_1 = a_2 = 0$), we have

$$\vec{h}^{(r)} = \vec{T} \wedge \vec{h},\ \rho_n^{(r)} = -\rho_g,\ \rho_g^{(r)} = \rho_n,\ K^{(r)}(s,0) = -\theta_g^2,\ H^{(r)}(s,0) = -\frac{\rho_g}{2}, \beta^{(r)} = c + \frac{\rho_n}{\rho_n^2 + \theta_g^2}\vec{h}.$$

corollary 5. *For* $a_3 = 1$ *(*$a_1 = a_2 = 0$*), the common curve* $c(s)$ *is a geodesic curve (resp. an asymptotic line) of the ruled surface* (2) *if and only if* $c(s)$ *is an asymptotic line (resp. geodesic curve) on* φ.

corollary 6. *For* $a_3 = 1$ *(*$a_1 = a_2 = 0$*), the following characterizations are satisfied:*

1. *The ruled surface* (2) *is developable if and only if* $c(s)$ *is a principal line on* φ.
2. *The ruled surface* (2) *is minimal along its base curve* $c(s)$ *if and only if* $c(s)$ *is a geodesic curve on* φ.
3. $c(s)$ *is the striction curve of* (2) *if and only if* $c(s)$ *is an asymptotic line on* φ.

Case 3: For $a_1 = 0$, $a_2 = a_3 = \frac{1}{\sqrt{2}}$, we have

$$\vec{h}^{(r)} = \frac{1}{\sqrt{2}}(\vec{T} \wedge \vec{h} + \vec{h}), \quad \rho_n^{(r)} = \frac{-\rho_g + \rho_n}{\sqrt{2}}, \quad \rho_g^{(r)} = \frac{\rho_n + \rho_g}{\sqrt{2}},$$
$$K^{(r)}(s,0) = -\theta_g^2, \quad H^{(r)}(s,0) = \frac{-\rho_g + \rho_n}{2\sqrt{2}}, \quad \beta^{(r)} = c + \frac{\rho_g + \rho_n}{(\rho_g + \rho_n)^2 + 2\theta_g^2}(\vec{g} + \vec{h}).$$

corollary 7. *For* $a_1 = 0$*,* $a_2 = a_3 = \frac{1}{\sqrt{2}}$ *the following characterizations are satisfied:*

1. *The ruled surface* (2) *is developable if and only if* $c(s)$ *is a principal line on* φ.
2. *The ruled surface* (2) *is minimal along its base curve* $c(s)$ *if and only if* $c(s)$ *is an asymptotic line on* (2).

4 Examples

In this section, we give three examples in the real euclidean 3-space $\mathbb{R}^3$, and for each example we present three cases with illustrations (Figures 1, 2, and 3) of the regular surface φ (in yellow) and the constructed ruled surface Ψ (in blue).

Example 1

Consider the hyperboloid of one sheet as a regular surface parameterized by

$$\varphi(u,t) = (\cos u - \frac{t}{\sqrt{2}}\sin u, \sin u + \frac{t}{\sqrt{2}}\cos u, \frac{t}{\sqrt{2}}). \tag{9}$$

Since the unit circle $c(s) = (\cos s, \sin s, 0)$ lies on the surface (9), the Darboux frame vectors of $c(s)$ on the hyperboloid (9) are given respectively by

$$\vec{T}(s) = \begin{pmatrix} -\sin s \\ \cos s \\ 0 \end{pmatrix}, \quad \vec{g}(s) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{h}(s) = \begin{pmatrix} \cos s \\ \sin s \\ 0 \end{pmatrix}. \tag{10}$$

The normal curvature, the geodesic curvature, and the geodesic torsion of $c = c(s)$ on the hyperboloid (9) take the following values, respectively

$$\rho_n = -1, \ \rho_g = \theta_g = 0.$$

Thus, the studied ruled surface which is constructed with the unit circle $c(s) = (\cos s, \sin s, 0)$ and Darboux frame vectors (10) is parametrized by

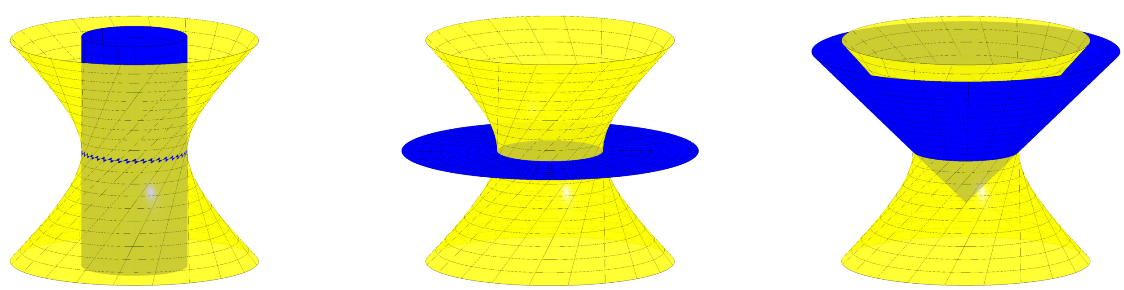
$$\Psi(s, v) = (\cos s, \sin s, 0) + v \begin{pmatrix} -a_1 \sin s + a_3 \cos s \\ a_1 \cos s + a_3 \sin s \\ a_2 \end{pmatrix}. \tag{11}$$

We have

$$\rho_n^{(r)} = -\frac{a_2}{\sqrt{a_2^2 + a_3^2}}, \quad \rho_g^{(r)} = -\frac{a_3}{\sqrt{a_2^2 + a_3^2}}, \quad \theta_g^{(r)} = 0,$$
$$K^{(r)}(s, 0) = -\frac{(a_1 a_2)^2}{(a_2^2 + a_3^2)^2}, \quad H^{(r)}(s, 0) = -\frac{a_2(1 - 2a_1^2)}{2(a_2^2 + a_3^2)^{3/2}}.$$

If $\|\vec{X}'\|^2 = a_1^2 + a_3^2 \neq 0$, then the ruled surface (11) is non-cylindrical and its striction curve takes the following form

$$\beta^{(r)}(s) = (\cos s, \sin s, 0) - \frac{a_3}{a_1^2 + a_3^2} \begin{pmatrix} -a_1 \sin s + a_3 \cos s \\ a_1 \cos s + a_3 \sin s \\ a_2 \end{pmatrix}.$$



$a_2 = 1, \ a_1 = a_3 = 0$ $\quad$ $a_3 = 1, \ a_1 = a_2 = 0$ $\quad$ $a_1 = 0, \ a_2 = a_3 = 1/\sqrt{2}$

Figure 1: Illustrations of the hyperboloid (9) and the ruled surface (11).

Example 2

Consider the torus parameterized by

$$\varphi(u,t) = ((2+\cos u)\cos t, (2+\cos u)\sin t, \sin u). \tag{12}$$

The Darboux frame vectors of the unit circle $c(s) = (2+\cos s, 0, \sin s)$ which lies on the torus (12), are

$$\vec{T}(s) = \begin{pmatrix} -\sin s \\ 0 \\ \cos s \end{pmatrix}, \quad \vec{g}(s) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{h}(s) = \begin{pmatrix} -\cos s \\ 0 \\ -\sin s \end{pmatrix}. \tag{13}$$

We have

$$\rho_n = 1, \quad \rho_g = \theta_g = 0.$$

Thus, the ruled surface constructed with the unit circle $c(s) = (2+\cos s, 0, \sin s)$ and Darboux frame vectors (13) is represented by

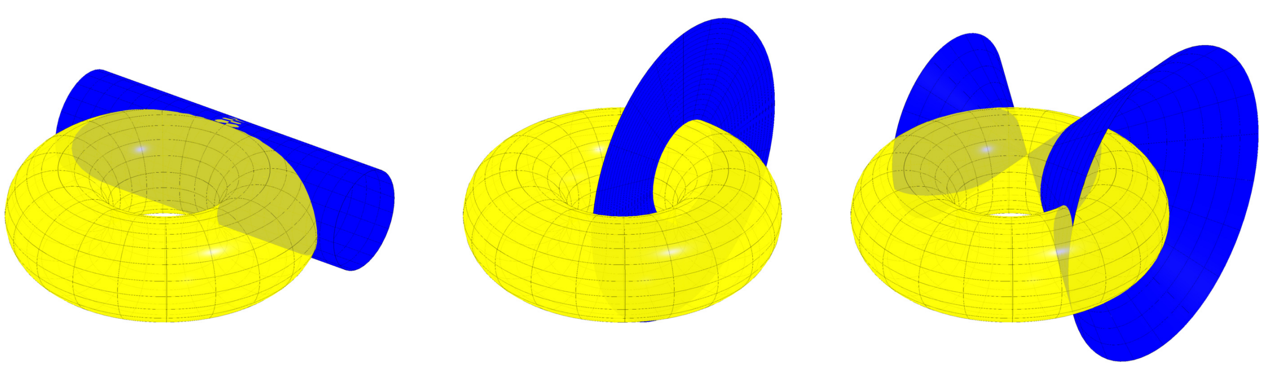
$$\Psi(s,v) = (2+\cos s, 0, \sin s) + v \begin{pmatrix} -a_1 \sin s - a_3 \cos s \\ a_2 \\ a_1 \cos s - a_3 \sin s \end{pmatrix}. \tag{14}$$

We have

$$\rho_n^{(r)} = \frac{a_2}{\sqrt{a_2^2 + a_3^2}}, \quad \rho_g^{(r)} = \frac{a_3}{\sqrt{a_2^2 + a_3^2}}, \quad \theta_g^{(r)} = 0,$$
$$K^{(r)}(s,0) = -\frac{(a_2 a_1)^2}{(a_2^2 + a_3^2)^2}, \quad H^{(r)}(s,0) = \frac{a_2(1 - 2a_1^2)}{2(a_2^2 + a_3^2)^{3/2}}.$$

If $\|\vec{X}'\|^2 = a_1^2 + a_3^2 \neq 0$, then ruled surface (14) is non-cylindrical and its striction curve is given by

$$\beta^{(r)}(s) = (2+\cos s, 0, \sin s) + \frac{a_3}{a_3^2 + a_1^2} \begin{pmatrix} -a_1 \sin s - a_3 \cos s \\ a_2 \\ a_1 \cos s - a_3 \sin s \end{pmatrix}.$$



$a_2 = 1,\ a_1 = a_3 = 0$ $\qquad$ $a_3 = 1,\ a_1 = a_2 = 0$ $\qquad$ $a_1 = 0,\ a_2 = a_3 = 1/\sqrt{2}$

Figure 2: Illustrations of the Torus (12) and the ruled surface (14).

Example 3

Consider the regular surface parameterized by

$$\varphi(u,t) = \Big(2\cos\frac{u}{2} - \frac{1}{\sqrt{2}}t\sin\frac{u}{2}, 2\sin\frac{u}{2} - \frac{1}{\sqrt{2}}t\cos\frac{u}{2}, \frac{1}{\sqrt{2}}t\Big). \tag{15}$$

Darboux frame vectors of the curve $c(s) = (2\cos\frac{s}{2}, 2\sin\frac{s}{2}, 0)$, which lies on the regular surface (15), are

$$\vec{T}(s) = \begin{pmatrix} -\sin(\frac{s}{2}) \\ \cos(\frac{s}{2}) \\ 0 \end{pmatrix}, \quad \vec{g}(s) = \frac{1}{\sqrt{1+\sin^2(s)}} \begin{pmatrix} -\sin(s)\cos(\frac{s}{2}) \\ -\sin(s)\sin(\frac{s}{2}) \\ 1 \end{pmatrix},$$
$$\vec{h}(s) = \frac{1}{\sqrt{1+\sin^2(s)}} \begin{pmatrix} \cos(\frac{s}{2}) \\ \sin(\frac{s}{2}) \\ \sin(s) \end{pmatrix}.$$

We have

$$\rho_n(s) = \frac{-1}{2\sqrt{1+\sin^2(s)}}, \quad \rho_g(s) = \frac{\sin(s)}{2\sqrt{1+\sin^2(s)}}, \quad \theta_g(s) = -\frac{\cos(s)}{1+\sin^2(s)}.$$

The constructed ruled surface is parametrized by

$$\Psi(s,v) \;=\; (2\cos(\tfrac{s}{2}), 2\sin(\tfrac{s}{2}), 0) \;+\; v \begin{pmatrix} -a_1\sin(\frac{s}{2}) + \dfrac{-a_2\sin(s)\cos(\frac{s}{2}) + a_3\cos(\frac{s}{2})}{\sqrt{1+\sin^2(s)}} \\ a_1\cos(\frac{s}{2}) + \dfrac{-a_2\sin(s)\sin(\frac{s}{2}) + a_3\sin(\frac{s}{2})}{\sqrt{1+\sin^2(s)}} \\ \dfrac{a_2 + a_3\sin(s)}{\sqrt{1+\sin^2(s)}} \end{pmatrix}. \tag{16}$$

We have

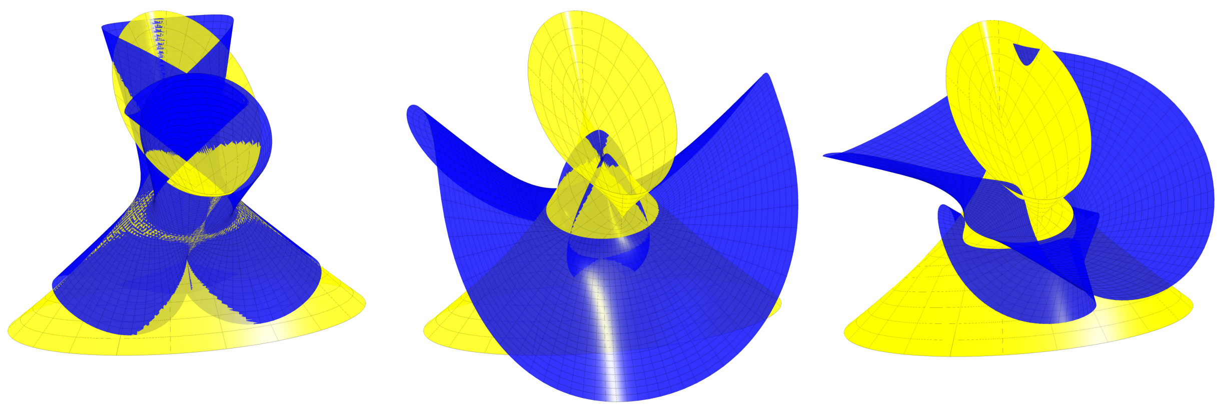
$$\rho_n^{(r)}(s) = -\frac{a_3\sin(s) + a_2}{2\sqrt{(a_3^2+a_2^2)(1+\sin^2(s))}}, \quad \rho_g^{(r)}(s) = \frac{-a_3 + a_2\sin(s)}{2\sqrt{(a_3^2+a_2^2)(1+\sin^2(s))}},$$
$$\theta_g^{(r)}(s) = -\frac{\cos(s)}{1+\sin^2(s)}, \quad K^{(r)}(s,0) = -\frac{[a_1(a_3\sin(s)+a_2)\sqrt{1+\sin^2(s)} + 2(a_2^2+a_3^2)\cos(s)]^2}{4(a_2^2+a_3^2)^2(1+\sin^2(s))^2},$$
$$H^{(r)}(s,0) = -\frac{[a_3(1-2a_1^2)\sin(s) + a_2(1-2a_1^2)]\sqrt{1+\sin^2(s)} - 4a_1(a_3^2+a_2^2)\cos(s)}{4(1+\sin^2(s))(a_2^2+a_3^2)^{3/2}}.$$

If

$$\|\vec{X}'(s)\|^2 = \frac{(a_2\sin(s)-a_3)^2(1+\sin^2(s)) + (a_1\sin(s)\sqrt{1+\sin^2(s)} + 2a_3\cos(s))^2}{4(1+\sin^2(s))^2}$$
$$+ \frac{(a_1\sqrt{1+\sin^2(s)} + 2a_2\cos(s))^2}{4(1+\sin^2(s))^2} \neq 0,$$

then the ruled surface (16) is non-cylindrical and its striction curve is given by:

$$\beta^{(r)}(s) = (2\cos(\tfrac{s}{2}), 2\sin(\tfrac{s}{2}), 0) + \frac{a_1}{\|\vec{X}'\|^2}\begin{pmatrix} -a_1\sin(\frac{s}{2}) + \dfrac{-a_2\sin(s)\cos(\frac{s}{2}) + a_3\cos(\frac{s}{2})}{\sqrt{1+\sin^2(s)}} \\ a_1\cos(\frac{s}{2}) + \dfrac{-a_2\sin(s)\sin(\frac{s}{2}) + a_3\sin(\frac{s}{2}}{\sqrt{1+\sin^2(s)}} \\ \dfrac{a_2 + a_3\sin(s)}{\sqrt{1+\sin^2(s)}} \end{pmatrix}.$$



$a_2 = 1,\ a_1 = a_3 = 0$ $a_3 = 1,\ a_1 = a_2 = 0$ $a_1 = 1/\sqrt{2},\ a_2 = a_3 = 1/2$

Figure 3: Illustrations of the surface (15) and the ruled surface (16)

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