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NEW CHARACTERIZATIONS FOR SPECIAL CURVES USING THE SPHERICAL INDICATRICES

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Abstract

In this paper, we determine the relation between the Frenet-Serret invariants of γ and the Frenet-Serret curvatures of the spherical indicatrix of tangent vector, principal normal vector, and Bishop Darboux vector to give a new characterization of a general helix and a slant helix. Additionally, the new study of Bishop Darboux spherical indicatrix is given. Finally, we have presented some examples that are computed in detail.

1. Introduction

In differential geometry, a general helix in Euclidean 3-space E^3 is defined by the property that tangent vector makes a constant angle with a fixed direction. A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant.

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In [7], a slant helix in Euclidean space E^3 was defined by the property that the principal normal vector makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi [7] showed that γ is a slant helix in E^3, if and only if the geodesic curvature

$$\sigma(s) = \left[\frac{\kappa^2}{(\kappa^2 + \tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right],$$

of the principal normal indicatrix of a space curve γ is a constant function. In [5], the authors have studied spherical images of tangent vector and binormal vector of a slant helix and they showed that the spherical images are spherical helix.

The new spherical images of regular curve, which are called Bishop spherical images are determined by using Bishop frame vectors in [6].

2. Preliminaries

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. The curve $\alpha : I \subset R \to E^3$ is called a unit speed curve if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ for each $s \in I$. $T(s) = \alpha'(s)$ is a unit tangent vector of α and $\kappa(s) = \|\alpha''(s)\|$ is the curvature of α at s. If $\kappa(s) \neq 0$, then the unit principal normal vector $N(s)$ of the curve α at s is given by $\alpha'' = \kappa(s)N(s)$. The unit vector $B(s) = T(s) \times N(s)$ is called the unit binormal vector of α at s. The Frenet-Serret formulae are given in [2] written under matrix form

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where

$$\langle T, T \rangle = \langle N, N \rangle = \langle B, B \rangle = 1,$$

$$\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0,$$

and

$$\tau = -\langle B', N \rangle.$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used (for details, see [1]). The Bishop frame is expressed as [1, 3]

$$\begin{bmatrix} T' \\ M_1' \\ M_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}.$$

Here, we shall call the set $\{T, M_1, M_2\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures. The relation matrix may be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & \sin\theta(s) \\ 0 & -\sin\theta(s) & \cos\theta(s) \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix},$$

where

$$\theta(s) = \arctan \frac{k_2}{k_1}, \tag{1}$$

$$\tau(s) = \theta'(s), \tag{2}$$

and

$$\kappa(s) = \sqrt{k_1^2 + k_2^2}. \tag{3}$$

Here, Bishop curvatures are defined by

$$\begin{cases} k_1 = \kappa \cos \theta(s), \\ k_2 = \kappa \sin \theta(s). \end{cases}$$

Yilmaz et al. [6] investigated spherical images of a regular curve which correspond to each vector fields of the Bishop frame.

It is well known that for a unit speed curve with non-vanishing curvatures, the following definition and result hold [6]:

Definition 2.1. Let $\gamma = \gamma(s)$ be a regular curve in E^3. If we translate of the first vector field (tangent) of Bishop frame to the center O of unit sphere in the E^3, we obtain a spherical image $\xi = \xi(s_\xi)$. This curve is called *tangent Bishop spherical image* or indicatrix of curve $\gamma = \gamma(s)$.

The first curvature κ_ξ of $\xi = \xi(s_\xi)$ is given in [6] by

$$\kappa_\xi = \|\dot{T}_\xi\| = \sqrt{1 + \left[\frac{k_2^3}{(k_1^2 + k_2^2)^2}\left(\frac{k_1}{k_2}\right)'\right]^2 + \left[\frac{k_1^3}{(k_1^2 + k_2^2)^2}\left(\frac{k_2}{k_1}\right)'\right]^2}. \tag{4}$$

As it is indicated in [3], if a rigid body moves along a regular curve, then the motion of body consists of translation and rotation along the curve α. The rotation is determined by an angular velocity vector ω, which is called the *Bishop Darboux vector* and given by

$$\omega = -k_2 M_1 + k_1 M_2,$$

ω satisfies these equations

$$T' = \omega \times T,$$

$$M_1' = \omega \times M_1,$$

$$M_2' = \omega \times M_2,$$

if we unitize the Bishop Darboux vector, we get

$$C = \frac{\omega}{\|\omega\|} = \frac{-k_2 M_1 + k_1 M_2}{\sqrt{k_1^2 + k_2^2}}.$$

3. Main Results

3.1. A new characterization of general helix according to tangent vector field of Bishop frame

Theorem 3.1. *Let $\varphi = \varphi(s)$ be a regular curve with curvatures κ and τ. If φ lies on the surface of sphere, then*

$$\tau^2(r^2\kappa^2 - 1) = \left(\frac{\kappa'}{\kappa}\right)^2, \tag{5}$$

where r = constant is the radius of sphere.

Proof 3.2. To prove that the condition is necessary, we differentiate

$$\|\varphi(s)\|^2 = r^2,$$

we obtain

$$\langle \varphi(s), T \rangle = 0.$$

By differentiating, we have

$$\langle \varphi(s), N \rangle = -\frac{1}{\kappa}, \tag{6}$$

since

$$\left(\frac{1}{\kappa}\right)^2 + (\langle\varphi(s),\ B\rangle)^2 = r^2. \qquad (7)$$

So, by differentiating of the formula (6), we get

$$(\langle\varphi(s),\ B\rangle)\,\tau = \frac{\kappa'}{\kappa^2}.$$

Take a square of this result and using (7), we have

$$\left(r^2 - \left(\frac{1}{\kappa}\right)^2\right)\tau^2 = \left(\frac{\kappa'}{\kappa^2}\right)^2.$$

By multiplying by κ^2, we get the equation as desired.

Corollary 3.3. *Let* $\varphi = \varphi(s)$ *be a curve lies on the surface of radius* r *with the curvatures* κ *and* τ. κ *is a constant function if and only if* $\tau = 0$.

In another words, any spherical curve with $\kappa =$ *constant* (*resp.,* $\tau = 0$) *is a circle.*

Proof 3.4. The sufficiency condition is clear. For to prove that the condition is necessary, we suppose that $\kappa =$ constant. Two situations are distingue

(1) If $\kappa = \text{constant} \neq \frac{1}{r}$, then $\tau = 0$ (by Equation (5)).

(2) If $\kappa = \frac{1}{r}$, the Equation (7) became

$$(\langle\varphi(s),\ B\rangle)^2 = 0.$$

Differentiate this equation and using the Equation (6), we obtain

$$\left\langle\frac{d\varphi}{ds},\ B\right\rangle + \left\langle\varphi(s),\ \frac{dB}{ds}\right\rangle = \frac{\tau}{\kappa} = 0.$$

Since, we immediately arrive at

$$\tau = 0.$$

Theorem 3.5. *Let* $\xi = \xi(s_\xi)$ *be a spherical indicatrix of tangent vector of* γ *regular curve. There exists a relation among Frenet-Serret invariants of* $\xi = \xi(s_\xi)$ *and the Frenet-Serret invariants of* $\gamma = \gamma(s)$ *as follows*:

$$\kappa_\xi = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}. \tag{8}$$

Proof 3.6. Let $\xi = \xi(s_\xi)$ be a spherical indicatrix of the tangent vector of γ regular curve. From (4), we have

$$\kappa_\xi = \sqrt{1 + \left[\frac{k_2}{\sqrt{k_1^2 + k_2^2}} \frac{k_2^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}} \left(\frac{k_1}{k_2}\right)'\right]^2 + \left[\frac{k_1}{\sqrt{k_1^2 + k_2^2}} \frac{k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}} \left(\frac{k_2}{k_1}\right)'\right]^2}.$$

Using (1) in (2), we have

$$\tau(s) = \left(\arctan\left(\frac{k_2}{k_1}\right)\right)'$$

$$= \frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{k_1^2 + k_2^2}.$$

By the formula (3) of curvature, we write

$$\frac{\tau}{\kappa} = \frac{k_1^2\left(\frac{k_2}{k_1}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} = -\frac{k_2^2\left(\frac{k_1}{k_2}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}}. \tag{9}$$

We have immediately

$$\kappa_\xi = \sqrt{1 + \left[\frac{k_2}{\sqrt{k_1^2 + k_2^2}} \frac{\tau}{\kappa}\right]^2 + \left[\frac{k_1}{\sqrt{k_1^2 + k_2^2}} \frac{\tau}{\kappa}\right]^2} = \sqrt{1 + (\frac{\tau}{\kappa})^2}.$$

Corollary 3.7. *Let* $\xi = \xi(s_\xi)$ *be the tangent Bishop spherical image of the curve* $\gamma = \gamma(s)$. γ *is a general helix if and only if* $\xi = \xi(s_\xi)$ *is a circle.*

Proof 3.8. If γ is a general helix, then

$$\frac{\tau}{\kappa} = \text{constant}.$$

By Theorem 3.5,

$$\kappa_\xi = \text{constant}.$$

By the Corollary 3.3, we deduce that the ξ is a circle.

Inversely, if ξ is a circle, using the Equation (8) and the property of a general helix, we conclude that γ is a general helix.

3.2. A new characterization of a slant helix according to principal normal indicatrix

In this section, we give a characterization for a unit speed curve γ in E^3 to be a slant helix by using its principal normal indicatrix (N).

The following remark is given in [4]:

Remark 3.9. If the Frenet frame of the principal normal indicatrix N of a curve γ is $\{T_N, N_N, B_N\}$, then we have the first curvature of Frenet frame

$$\kappa_N = \frac{\sqrt{(\kappa^2 + \tau^2)^3 + (\kappa\tau' - \kappa'\tau)^2}}{(\kappa^2 + \tau^2)^{\frac{3}{2}}},$$

from this, the curvature of principal normal indicatrix (N) is

$$\kappa_N = \sqrt{1 + \sigma^2(s)}, \tag{10}$$

(see [4]).

Using the result of the Corollary 3.3 and the expression (10), it is easy to state the following theorem:

Theorem 3.10. *$\gamma = \gamma(s)$ is a slant helix according to Frenet frame if and only if the principal normal spherical image is a circle.*

3.3. A new characterization of general helix according to Bishop Darboux spherical indicatrix

As indicated in [3], another Bishop spherical image is determined in order to studies this curve, we determine its Frenet-Serret invariants in terms of Frenet invariants and Bishop invariants of γ.

Definition 3.11. Let $\gamma = \gamma(s)$ be a regular curve in E^3. If we translate of the Bishop Darboux vector field $C = \frac{\vec{\omega}}{\|\omega\|}$ to the center O of unit sphere, we obtain a spherical image $\alpha_C = \alpha_C(s_{\alpha_C})$. This curve is called *Bishop Darboux spherical image* or indicatrix of curve $\gamma = \gamma(s)$.

One can differentiate of α_C with respect to s

$$\alpha'_C = \frac{d\alpha_C}{ds_C}\frac{ds_C}{ds} = C'(s).$$

Here, we shall denote differentiation according to s by dash, and differentiation according to s_C by dot.

So, we have

$$\dot{\alpha}_C \frac{ds_C}{ds} = C'. \tag{11}$$

In [3], the authors have expressed C', in the terms of Bishop frame vector fields, as follows:

$$C' = \frac{k_2k_1k'_1 - k'_2k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}} M_1 + \frac{k'_1k_2^2 - k_1k_2k'_2}{(k_1^2 + k_2^2)^{\frac{3}{2}}} M_2$$

$$= \frac{k_1k_2^2\left(\frac{k_1}{k_2}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} M_1 + \frac{k_2^3\left(\frac{k_1}{k_2}\right)'}{(k_1^2 + k_2^2)^{\frac{3}{2}}} M_2.$$

Using the Equation (9), we may write

$$C' = -k_1 \frac{\tau}{\kappa} M_1 - k_2 \frac{\tau}{\kappa} M_2.$$

We have the tangent vector of spherical image as follows:

$$T_{\alpha_C} = -\frac{k_1}{\kappa} M_1 - \frac{k_2}{\kappa} M_2 = -\frac{1}{\kappa}(k_1 M_1 + k_2 M_2) = -N,$$

where

$$\frac{ds_C}{ds} = \tau.$$

Or, using the Equation (3), we can write

$$T_{\alpha_C} = -\frac{1}{\sqrt{1+\left(\frac{k_2}{k_1}\right)^2}} M_1 - \frac{1}{\sqrt{1+\left(\frac{k_1}{k_2}\right)^2}} M_2,$$

where

$$\frac{ds_C}{ds} = -\frac{k_2^2\left(\frac{k_1}{k_2}\right)'}{k_1^2 + k_2^2} = \tau.$$

In order to determine the first curvature of α_C, one can calculate

$$T'_{\alpha_C} = \dot{T}_{\alpha_C} \frac{ds_C}{ds} = -N' = \kappa T - \tau B,$$

since, we have

$$\dot{T}_{\alpha_C} = \frac{\kappa}{\tau} T - B,$$

since, we express

$$\kappa_{\alpha_C} = \|\dot{T}_{\alpha_C}\| = \sqrt{1+\left(\frac{\kappa}{\tau}\right)^2},$$

using (9), we will also have

$$\kappa_{\alpha_C} = \sqrt{1 + \left[\frac{(k_1^2 + k_2^2)^{\frac{3}{2}}}{k_1^2\left(\frac{k_2}{k_1}\right)'}\right]^2}.$$

Therefore, we have the principal normal

$$N_{\alpha_C} = \frac{\dot{T}_{\alpha_C}}{\|\dot{T}_{\alpha_C}\|} = \frac{1}{\kappa_{\alpha_C}}\left[\frac{\kappa}{\tau}T - B\right].$$

From the relation matrix, Equations (3) and (9) we have

$$N_{\alpha_C} = \frac{1}{k_{\alpha_C}}\left[\frac{(k_1^2 + k_2^2)^{\frac{3}{2}}}{k_1^2\left(\frac{k_2}{k_1}\right)'}T + \frac{k_2}{\sqrt{k_1^2 + k_2^2}}M_1 - \frac{k_1}{\sqrt{k_1^2 + k_2^2}}M_2\right].$$

By the cross product of $T_{\alpha_C} \times N_{\alpha_C}$, we obtain the binormal vector field

$$B_{\alpha_C} = \frac{1}{\kappa_{\alpha_C}}\left[T + \frac{\kappa}{\tau}B\right].$$

On the other hand, we can express B_{α_C} in term of Bishop invariants as follows:

$$B_{\alpha_C} = \frac{1}{\kappa_{\alpha_C}}\left[T - \frac{k_2(k_1^2 + k_2^2)}{k_1^2\left(\frac{k_2}{k_1}\right)'}M_1 + \frac{k_1^2 + k_2^2}{k_1\left(\frac{k_2}{k_1}\right)'}M_2\right].$$

In order to determinate the torsion, we differentiate B_{α_C} and we write

$$B'_{\alpha_C} = \dot{B}_{\alpha_C}\frac{ds_C}{ds} = -\frac{\left(\frac{\kappa}{\tau}\right)\left(\frac{\kappa}{\tau}\right)'}{\left(1 + \left(\frac{\kappa}{\tau}\right)^2\right)^{\frac{3}{2}}}T + \frac{\left(\frac{\kappa}{\tau}\right)'}{\left(1 + \left(\frac{\kappa}{\tau}\right)^2\right)^{\frac{3}{2}}}B,$$

since, we have

$$\dot{B}_{\alpha_C} = -\frac{1}{\tau}\frac{\left(\frac{\kappa}{\tau}\right)\left(\frac{\kappa}{\tau}\right)'}{\left(1+\left(\frac{\kappa}{\tau}\right)^2\right)^{\frac{3}{2}}}T + \frac{1}{\tau}\frac{\left(\frac{\kappa}{\tau}\right)'}{\left(1+\left(\frac{\kappa}{\tau}\right)^2\right)^{\frac{3}{2}}}B.$$

By the formulae of the torsion, we have

$$\tau_{\alpha_C} = -\left\langle \dot{B}_{\alpha_C}, N_{\alpha_C}\right\rangle$$
$$= \frac{1}{\tau}\left[\frac{\left(\frac{\kappa}{\tau}\right)'}{1+\left(\frac{\kappa}{\tau}\right)^2}\right],$$

using (9), we will also have

$$\tau_{\alpha_C} = \frac{1}{\tau}\frac{\left[\frac{(k_1^2+k_2^2)^{\frac{3}{2}}}{k_1^2\left(\frac{k_2}{k_1}\right)'}\right]'}{1+\left[\frac{(k_1^2+k_2^2)^{\frac{3}{2}}}{k_1^2\left(\frac{k_2}{k_1}\right)'}\right]^2}.$$

Theorem 3.12. *Let* $\alpha_C = \alpha_C\left(s_{\alpha_C}\right)$ *be Bishop Darboux spherical image of* $\gamma = \gamma(s)$. γ *is a general helix if and only if* α_C *is a circle.*

Proof 3.13. Similar to proof of the Corollary 3.7, the above result can be obtained by the equation

$$\kappa_{\alpha_C} = \sqrt{1+\left(\frac{\kappa}{\tau}\right)^2},$$

property of general helix and Corollary 3.3.

4. Examples

Example 4.1. Let us consider a unit speed circular helix by

$$\beta = \beta(s) = \left(a \cos \frac{s}{c},\ a \sin \frac{s}{c},\ \frac{bs}{c}\right),$$

where $c = \sqrt{a^2 + b^2} \in \mathbb{R}$. One can calculate its Frenet-Serret apparatus as the following:

$$\begin{cases} \kappa = \dfrac{a}{c^2}, \\ \tau = \dfrac{b}{c^2}, \\ T = \dfrac{1}{c}\left(- a \sin \dfrac{s}{c},\ a \cos \dfrac{s}{c},\ b\right), \\ N = \left(- \cos \dfrac{s}{c},\ - \sin \dfrac{s}{c},\ 0\right), \\ B = \dfrac{1}{c}\left(b \sin \dfrac{s}{c},\ - b \cos \dfrac{s}{c},\ a\right). \end{cases}$$

In order to determine the Bishop frame of the curve $\beta = \beta(s)$, let us from

$$\theta(s) = \int_0^s \frac{b}{c^2}\, ds = \frac{bs}{c^2}.$$

Since, we can write the transformation matrix

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{bs}{c^2} & \sin \frac{bs}{c^2} \\ 0 & - \sin \frac{bs}{c^2} & \cos \frac{bs}{c^2} \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}.$$

By the method of Cramer, one can obtain the Bishop trihedra as follows:

The tangent:

$$T = \frac{1}{c}\left(- a \sin \frac{s}{c},\ a \cos \frac{s}{c},\ b\right).$$

The M_1:

$$M_1 = \left(-\cos\frac{s}{c}\cos\frac{bs}{c^2} - \frac{b}{c}\sin\frac{s}{c}\sin\frac{bs}{c^2},\ \frac{b}{c}\cos\frac{s}{c}\sin\frac{bs}{c^2} - \sin\frac{s}{c}\cos\frac{bs}{c^2},\ -\frac{a}{c}\sin\frac{bs}{c^2}\right).$$

The M_2:

$$M_2 = \left(\frac{b}{c}\sin\frac{s}{c}\cos\frac{bs}{c^2} - \cos\frac{s}{c}\sin\frac{bs}{c^2},\ -\frac{b}{c}\cos\frac{s}{c}\cos\frac{bs}{c^2} - \sin\frac{s}{c}\sin\frac{bs}{c^2},\ -\frac{a}{c}\cos\frac{bs}{c^2}\right).$$

So, one can calculate the Bishop Darboux field as follows:

$$C = \left(\frac{b}{c}\sin\left(\frac{s}{c}\right),\ -\frac{b}{c}\cos\left(\frac{s}{c}\right),\ \frac{a}{c}\right).$$

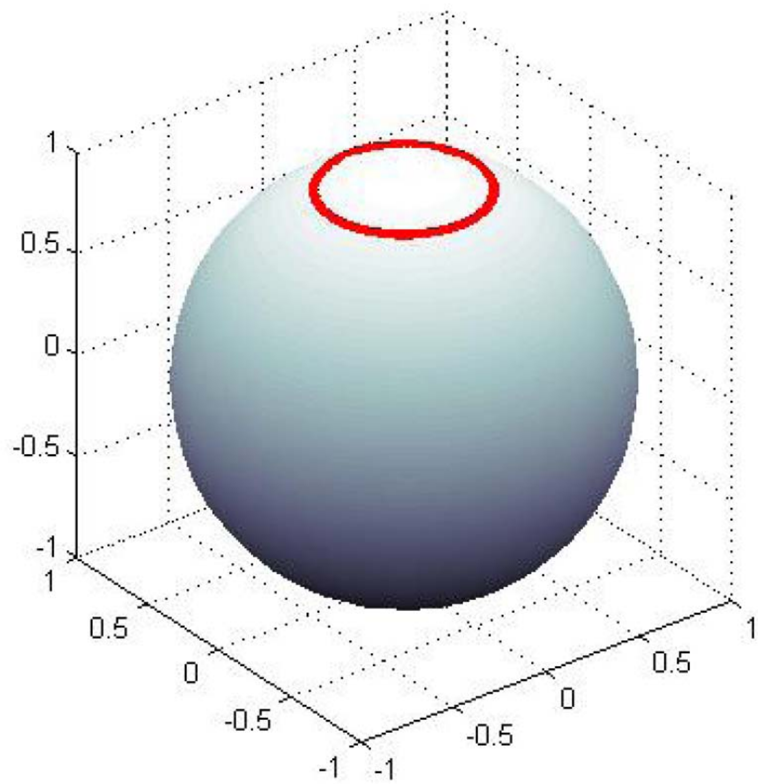

Figure 1. Bishop Darboux spherical indicatrix of $\beta = \beta(s)$.

Example 4.2. Let us consider the following unit speed curve $\gamma = (\gamma_1, \gamma_2, \gamma_3)$:

$$\begin{cases} \gamma_1 = \dfrac{9}{208}\sin(16s) - \dfrac{1}{117}\sin(36s), \\ \gamma_2 = -\dfrac{9}{208}\cos(16s) + \dfrac{1}{117}\cos(36s), \\ \gamma_3 = \dfrac{6}{65}\sin(10s), \end{cases}$$

this curvature's functions are expressed in [6] as

$$\begin{cases} \kappa(s) = -24\sin(10s), \\ \tau = 24\cos(10s). \end{cases}$$

The vectors of Frenet-Serret frame along the curve γ are given by

The tangent:

$$T = \begin{cases} \gamma_1' = \dfrac{9}{13}\cos(16s) - \dfrac{4}{13}\cos(36s), \\ \gamma_2' = \dfrac{9}{13}\sin(16s) - \dfrac{4}{13}\sin(36s), \\ \gamma_3' = \dfrac{12}{13}\cos(10s). \end{cases}$$

The principal normal:

$$N = \begin{cases} N_1 = \dfrac{6}{13}\dfrac{\sin(16s)}{\sin(10s)} - \dfrac{6}{13}\dfrac{\sin(36s)}{\sin(10s)}, \\ N_2 = -\dfrac{6}{13}\dfrac{\cos(16s)}{\sin(10s)} + \dfrac{6}{13}\dfrac{\cos(36s)}{\sin(10s)}, \\ N_3 = \dfrac{5}{13}. \end{cases}$$

The binormal:

$$B = \begin{cases} B_1 = \dfrac{45}{169}\sin(16s) - \dfrac{20}{169}\sin(36s) + \dfrac{72}{169}\dfrac{\cos(16s)}{\tan(10s)} - \dfrac{72}{169}\dfrac{\cos(36s)}{\tan(10s)}, \\ B_2 = -\dfrac{45}{169}\cos(16s) + \dfrac{20}{169}\cos(36s) + \dfrac{72}{169}\dfrac{\sin(16s)}{\tan(10s)} - \dfrac{72}{169}\dfrac{\sin(36s)}{\tan(10s)}, \\ B_3 = -\dfrac{156}{169}\sin(10s). \end{cases}$$

We also need

$$\theta(s) = \int_0^s 24\cos(10s)ds = \frac{24}{10}\sin(10s).$$

The transformation matrix for the curve $\gamma = \gamma(s)$ has the form

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\frac{24}{10}\sin(10s)) & \sin(\frac{24}{10}\sin(10s)) \\ 0 & -\sin(\frac{24}{10}\sin(10s)) & \cos(\frac{24}{10}\sin(10s)) \end{bmatrix} \begin{bmatrix} T \\ M_1 \\ M_2 \end{bmatrix}.$$

By the method of Cramer, one can obtain the Bishop trihedra as follows:

The tangent:

$$T = \begin{cases} \dfrac{9}{13}\cos(16s) - \dfrac{4}{13}\cos(36s), \\ \dfrac{9}{13}\sin(16s) - \dfrac{4}{13}\sin(36s), \\ \dfrac{12}{13}\cos(10s). \end{cases}$$

The M_1:

$$M_1 = \cos(\frac{24}{10}\sin(10s))N - \sin(\frac{24}{10}\sin(10s))B.$$

The M_2:

$$M_2 = \sin(\frac{24}{10}\sin(10s))N + \cos(\frac{24}{10}\sin(10s))B.$$

Using the expression of C, the Bishop Darboux field, we have

$$C = -\sin\left(\frac{24}{10}\sin(10s)\right)M_1 + \cos\left(\frac{24}{10}\sin(10s)\right)M_2.$$

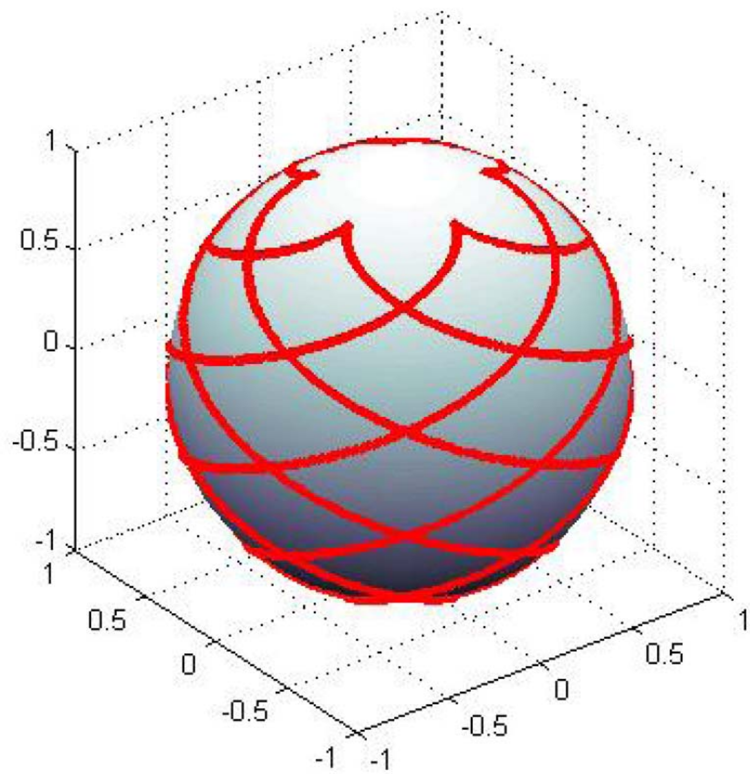

Figure 2. Bishop Darboux spherical indicatrix of $\gamma = \gamma(s)$.

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